1 Geometric interpretation of the SVD

In this exercise, we explore the geometric interpretation of symmetric matrices and how this connects to the SVD. We consider how a real 2×2 matrix acts on the unit circle, transforming it into an ellipse. It turns out that the principal semiaxes of the resulting ellipse are related to the singular values of the matrix, as well as the vectors in the SVD.

$$\underbrace{\begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{v_1} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{v_2} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} -\sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ -3\sqrt{2}/2 \end{bmatrix}$$

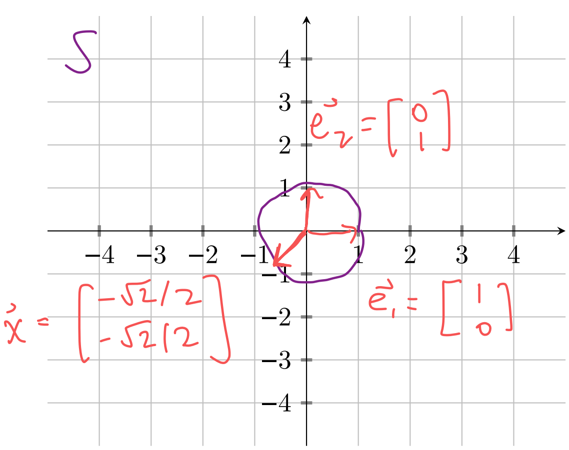
a) Consider the real 2×2 matrix

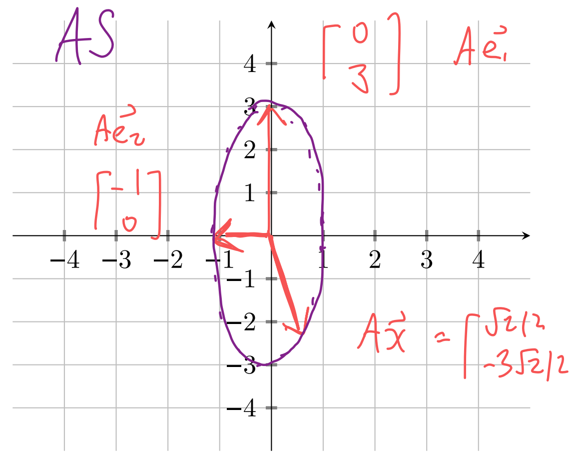
$$A = \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}.$$

Now consider the unit circle in $\mathbb{R}^2$,

$$S = \left\{ \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \mid 0 \le \theta < 2\pi \right\}.$$

Plot AS on the $\mathbb{R}^2$ plane.





$$\begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{bmatrix} -\sin\theta \\ 3\cos\theta \end{bmatrix}$$

$x = -\sin\theta$

$y = 3\cos\theta$

$\theta \in (0, 2\pi)$

b) Calculate the SVD of A. Write this as a matrix factorization, i.e. $A = U\Sigma V^\top$. full SVD

$$A = \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix}$$

0. Compute A^TA

$$\begin{bmatrix} 0 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$$

1. Compute eig of A^TA

$\lambda_1 = 9$ $\quad\quad$ $\lambda_2 = 1$

$A^TA - 9I = \begin{bmatrix} 0 & 0 \\ 0 & -8 \end{bmatrix}$ $\quad\quad$ $A^TA - I = \begin{bmatrix} 8 & 0 \\ 0 & 0 \end{bmatrix}$

$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad\quad$ $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

2. Compute $\sigma_i = \sqrt{\lambda_i}$

$\sigma_1 = \sqrt{9} = 3$ $\quad\quad$ $\sigma_2 = \sqrt{1} = 1$

3. Compute left-sing. vecs.

$u_i = \dfrac{Av_i}{\sigma_i}$

$u_1 = \frac{1}{3}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

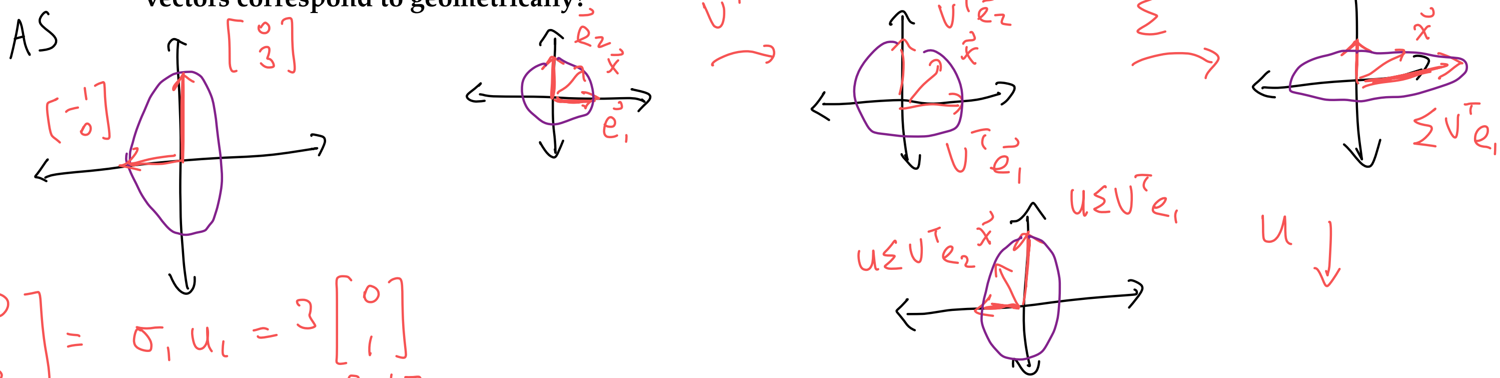
$u_2 = \frac{1}{1}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$$A = 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}^T + 1\begin{bmatrix} -1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}^T = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T}_{V^T}$$

$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

c) Consider the columns of the matrices U, V obtained in the previous part, and treat them as vectors in $\mathbb{R}^2$. Let $U = (\vec{u}_1\, \vec{u}_2)$, $V = (\vec{v}_1\, \vec{v}_2)$. Let σ_1, σ_2 be the singular values of A, where $\sigma_1 \geq \sigma_2$.

Draw in your plot of AS the vectors $\sigma_1\vec{u}_1$ and $\sigma_2\vec{u}_2$, drawn from the origin. What do these vectors correspond to geometrically?



$\begin{bmatrix}0\\3\end{bmatrix} = \sigma_1 u_1 = 3\begin{bmatrix}0\\1\end{bmatrix}$

$\begin{bmatrix}-1\\0\end{bmatrix} = \sigma_2 u_2 = 1\begin{bmatrix}-1\\0\end{bmatrix}$

d) **Repeat what you did above for the matrix $A = \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}$.** We can write an SVD for A as

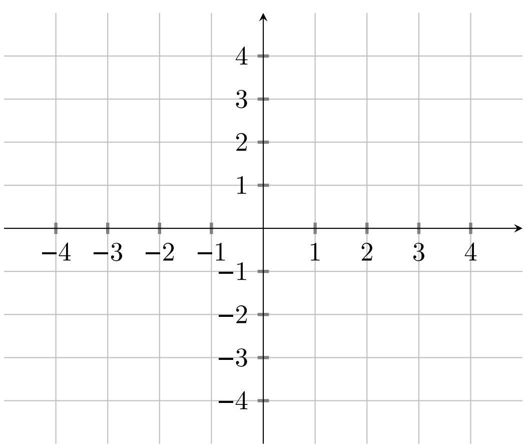
rotation by $-45°$

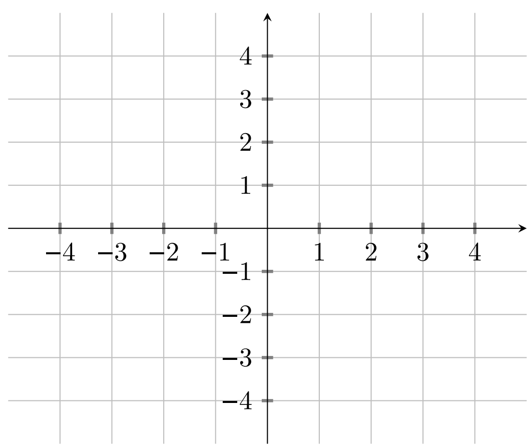
$$A = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$A = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$

$\begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$

$\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$





2 SVD and Induced 2-Norm

a) Show that if U is a unitary matrix then for any $\vec{x}$

$$\|U\vec{x}\| = \|\vec{x}\|.$$

$\|w\|^2 = \langle w, w \rangle$

$\|w\| = \sqrt{\langle w, w \rangle}$

$$\|U\vec{x}\| = \sqrt{\langle U\vec{x}, U\vec{x} \rangle}$$
$$= \sqrt{(U\vec{x})^T (U\vec{x})}$$

U is unitary so $U^T U = I$

$$= \sqrt{\vec{x}^T U^T U \vec{x}}$$
$$= \sqrt{\vec{x}^T \vec{x}}$$
$$= \|\vec{x}\|$$

b) Find the maximum

$$\max_{\{\vec{x}:\|\vec{x}\|=1\}} \|A\vec{x}\|$$

in terms of the singular values of A.

Induced Norm: if $y = Ax$ what is $\|y\|$ and what is $\max_x \|y\|$

Consider $\|A\vec{x}\|$, $A = U\Sigma V^T$

$$\|A\vec{x}\| = \|U\Sigma V^T \vec{x}\|$$

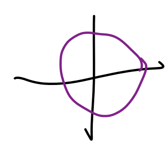
$V^T\vec{x}$ is a rotation of $\vec{x}$

$$= \|\Sigma V^T \vec{x}\|$$

$\vec{z} = V^T \vec{x} = V^{-1}\vec{x}$

$$= \|\Sigma \vec{z}\|$$

$$\Sigma = \begin{bmatrix} \sigma_1 & & & 0 \\ & \ddots & & \\ & & \sigma_r & \\ 0 & & & 0 \end{bmatrix}$$



max over all $\vec{x}$ with norm 1 $\|A\vec{x}\|$

max over all $\vec{z}$ with norm 1 $\|\Sigma \vec{z}\|$

$$\left\| \begin{bmatrix} \sigma_1 & & & 0 \\ & \ddots & \sigma_r & \\ 0 & & & 0 \end{bmatrix} \vec{z} \right\|$$

The optimal $\vec{z}$ will be:

$$\vec{z} = \begin{array}{c} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array} \begin{array}{c} z_1 \\ \vdots \\ z_n \end{array}$$

$$\left\| \Sigma \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\| = \sigma_1$$

Assume an opt. sol with any weight on $z_2, \ldots, z_n$

c) Find the $\vec{x}$ that maximizes the expression above.

$$\vec{z} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{e}_1$$

$$V\vec{z} = VV^T\vec{x}$$

$$\vec{x} = V\vec{z}$$

$VV^T = I$ since V is unitary

$$\vec{x} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \vec{v}_1$$