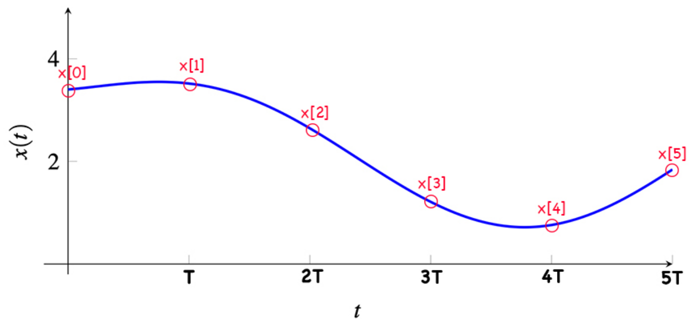
Sampling theorem

Let x be continuous signal bandlimited by frequency ω_{max}. We sample x with a period of T_s.



Given the discrete samples, we can try reconstructing the original signal f through sinc-interpolation where $\Phi(t) = \text{sinc}\left(\frac{t}{T_s}\right)$

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x[n]\Phi(t - nT_s)$$

We define the **sampling frequency** as $\omega_s = \frac{2\pi}{T_s}$. The Sampling Theorem says if $\omega_{max} < \frac{\pi}{T_s}$, or $\omega_s > 2\omega_{max}$, then we are able to recover the original signal, i.e. $x = \hat{x}$.

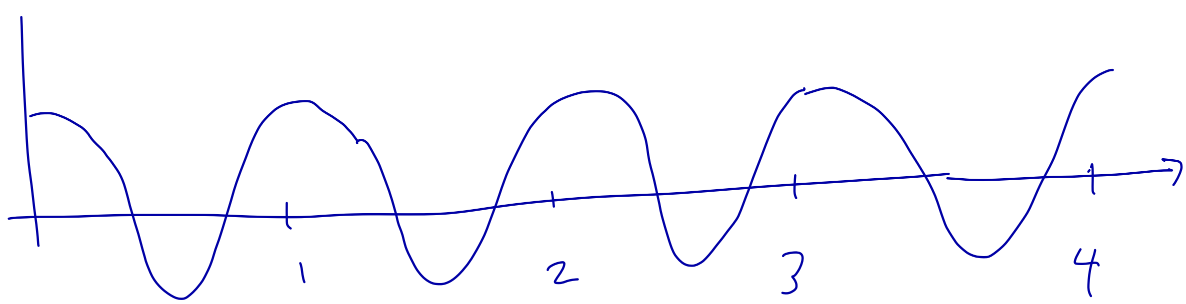
1 Sampling Theorem basics

Consider the following signal, $x(t)$ defined as,

$$x(t) = \cos(2\pi t). \tag{1}$$

$\omega = 2\pi$, period = 1

a) Sketch the signal $x(t)$, for $t \in [0, 4]$s.



b) Sketch discrete samples of $x(t)$ is the signal is sampled at a period of

i) $\frac{1}{4}$s

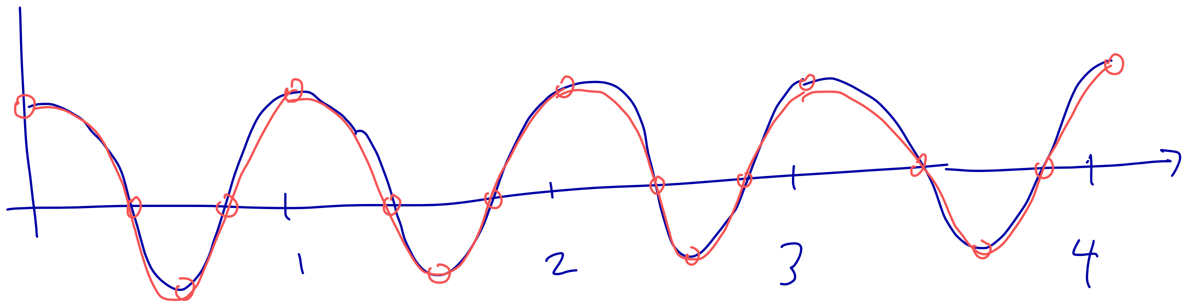
ii) $\frac{1}{2}$s

iii) 1s

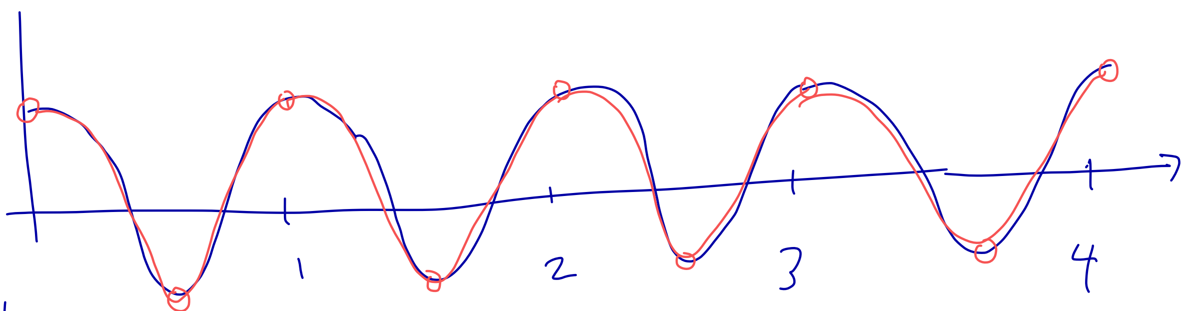
iv) 2s

How would you reconstruct a continuous signal $\hat{x}(t)$ if you only had the discrete samples for reconstruction?

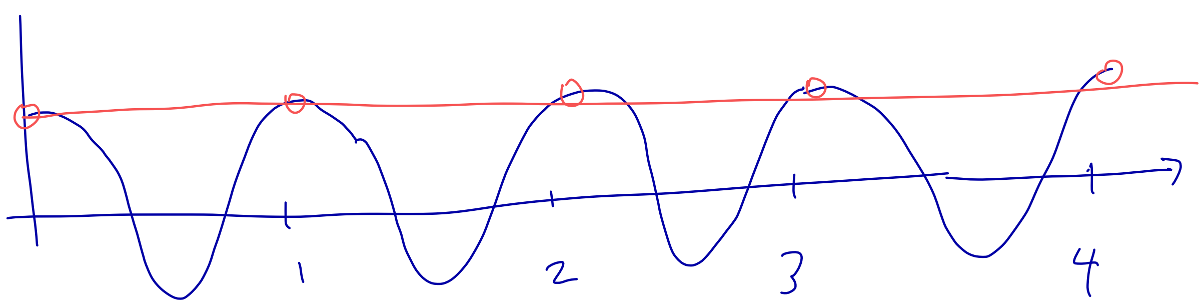
(i) $T_s = \frac{1}{4}$



(ii) $T_s = \frac{1}{2}$ $\omega_{max} = 2\pi$, $T < \frac{1}{2}$ for a perfect reconstr.



(iii) $T_s = 1$



$x[n] = 1$ for all n

c) What is the maximum frequency, $\omega_{\max}$, in radians per second? In Hertz?

$x(t) = \cos(2\pi t)$, $\omega_{max} = 2\pi \frac{rad}{s}$ or $f_{max} = 1\ Hz$

$f = \frac{\omega}{2\pi}$

d) If I sample every T seconds, what is the sampling frequency?

$\omega_s = \frac{2\pi}{T_s}$ $\quad T_s = T$, $\quad \boxed{\omega_s = \frac{2\pi}{T}}$

e) What is the smallest sampling period T that ~~would~~ may result in an imperfect reconstruction?

Sampling thm says $\omega_s > 2\omega_{max} = 4\pi$

$T = \frac{2\pi}{\omega_s}$ $\quad \frac{2\pi}{T} > 4\pi \rightarrow T < \frac{1}{2}$ for a perfect reconstr.

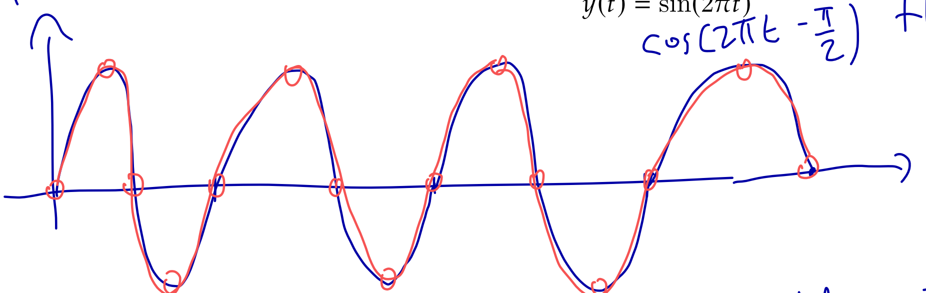
So if $T >= \frac{1}{2}$ we may have an imperfect reconstr. the smallest such T is $T = 1/2$.

f) Repeat part (b), for the signal

$$y(t) = \sin(2\pi t) \tag{2}$$

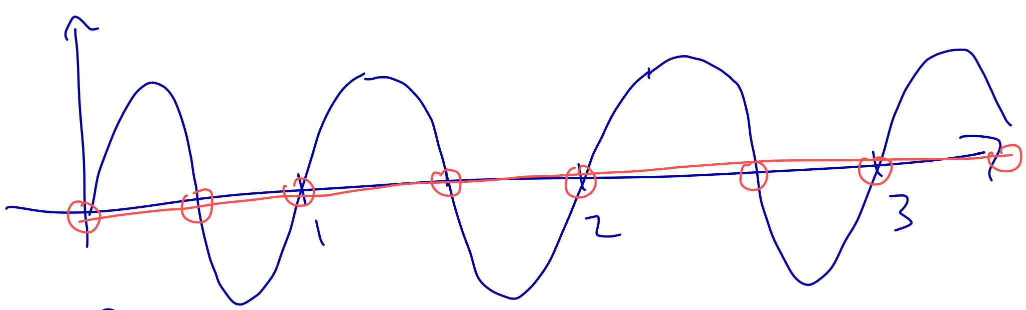
$\cos(2\pi t - \frac{\pi}{2})$

(i) $T = \frac{1}{4}$

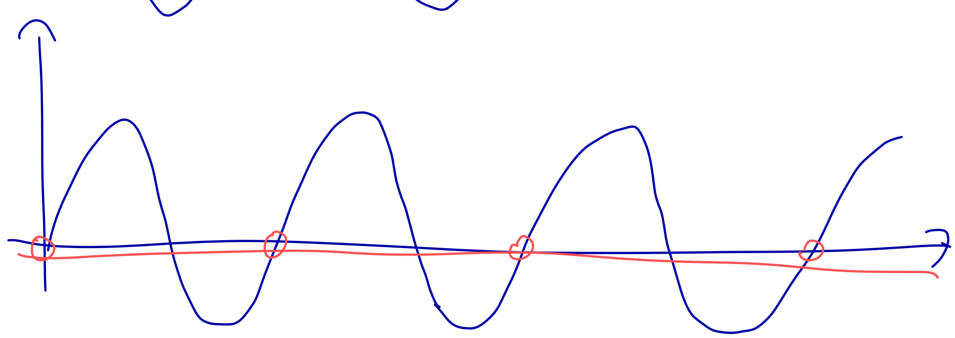


$\omega_{max} = 2\pi$, because $T = \frac{1}{2}$ we aren't guaranteed a perfect reconstr.

(ii) $T = 1/2$



(iv) $T = 1$



2 Aliasing

Consider the signal $x(t) = \sin(0.2\pi t)$.

a) At what period T should we sample so that sinc interpolation recovers a function that is identically zero?

$$\sin\left(\frac{\pi}{5} t\right)$$

$$T = 5k$$

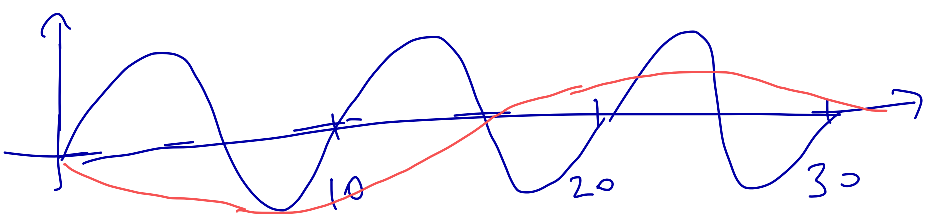
original signal: $x(t) = \sin\left(\frac{\pi}{5} t\right)$

sampled signal: $x[n] = \sin\left(\frac{\pi}{5} nT\right)$ for $n = 0, 1, 2, \ldots$

b) At what period T can we sample at so that sinc interpolation recovers the function $f(t) = -\sin\left(\frac{\pi}{15}t\right)$?

Samples to be $x[n] = -\sin\left(\frac{\pi}{15} \cdot n\right)$

aliased signal $x(t) = -\sin\left(\frac{\pi}{15} t\right)$



$$\sin x = -\cos\left(x + \frac{\pi}{2}\right)$$

$$x(t) = \sin\left(\frac{\pi}{5} t\right)$$

$$x[n] = \sin\left(\frac{\pi}{5} nT\right)$$

$$= \cos\left(\frac{\pi}{5} nT - \frac{\pi}{2}\right)$$

$$= \cos\left(2\pi n - \frac{\pi}{5} nT + \frac{\pi}{2}\right)$$

$$= \cos\left(\left(2\pi - \frac{\pi}{5} T\right) n + \frac{\pi}{2}\right)$$

$$= -\sin\left(\underbrace{\left(2\pi - \frac{\pi}{5} T\right)}_{\text{reconstructed freq}} n\right)$$

$$2\pi - \frac{\pi}{5} T = \frac{\pi}{15} T$$

$$\downarrow$$

$$T = 7.5$$