1 Diagonalization

Consider an $n \times n$ matrix A that has n linearly independent eigenvalue/eigenvector pairs $(\lambda_1, \vec{v}_1), \ldots, (\lambda_n, \vec{v}_n)$ that can be put into a matrices V and Λ.

$$V = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \quad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$A\vec{v}_i = \lambda_i \vec{v}_i$

eigenvector property

Important Trick

a) Show that $AV = V\Lambda$.

$$AV = A \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ A\vec{v}_1 & \cdots & A\vec{v}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1\vec{v}_1 & \cdots & \lambda_n\vec{v}_n \\ | & & | \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1\vec{a}_1 + \ldots + x_n\vec{a}_n$$

$$V \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Look at $V \cdot \begin{bmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ← first col of Λ

$$\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lambda_1 \vec{v}_1 + 0 \cdot \vec{v}_2 + \ldots + 0 \cdot \vec{v}_n$$

$$= \lambda_1 \vec{v}_1$$

$$V \vec{\lambda}_i = \lambda_i \vec{v}_i$$

↑ i^{th} col of Λ

$$V\Lambda = \begin{bmatrix} | & & | \\ \lambda_1\vec{v}_1 & \cdots & \lambda_n\vec{v}_n \\ | & & | \end{bmatrix}$$

b) Use the fact in part (a) to conclude that $\underline{A = V\Lambda V^{-1}}$.

$$AV = V\Lambda$$

Because A has n linearly independent eigenvectors,

$V = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$ must be invertible ← has L.I. columns & is square

$$AV = V\Lambda$$

$$AVV^{-1} = V\Lambda V^{-1}$$

$$A = V\Lambda V^{-1}$$

2 Systems of Differential Equations

Consider a system of differential equations (valid for $t \geq 0$)

$$\frac{d}{dt}x_1(t) = -4x_1(t) + x_2(t) \tag{1}$$

x_1, x_2 states

$$\frac{d}{dt}x_2(t) = 2x_1(t) - 3x_2(t) \tag{2}$$

with initial conditions $x_1(0) = 3$ and $x_2(0) = 3$.

a) Write out the system of differential equations and initial conditions in the matrix/vector form

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) \tag{3}$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}}$$

$$\vec{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

b) Find the eigenvalues λ_1, λ_2 and eigenspaces for the differential matrix A.

$$\det(A - \lambda I) = 0$$

$$\lambda^2 + 7\lambda + 10 = 0 \rightarrow \lambda = -2, -5$$

Eigenspaces

$$A - (-5I) = A + 5I = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \qquad \vec{v}_1 = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_1 = -5$$

$$A - (-2I) = A + 2I = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \qquad \vec{v}_2 = \beta \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\lambda_2 = -2$$

c) Let us define a new variable $\vec{z} = V^{-1}\vec{x}$. Use the diagonalization of $A = V\Lambda V^{-1}$ to rewrite the original differential equation in terms of $z_i(t)$ and a diagonal matrix Λ.

$$\frac{d}{dt}\vec{z}(t) = \Lambda\vec{z}(t) \tag{4}$$

Remember to find the new initial conditions $z_1(0), z_2(0)$.

$\checkmark$ $A = \underline{V \Lambda V^{-1}}$

$V = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$

$V^{-1} = \begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{bmatrix}$

$V^{-1} = \frac{1}{\det(V)} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$

$\det(V) = 3$

Start: $\frac{d}{dt}\vec{x} = A\vec{x}$

$\frac{d}{dt}\vec{x} = V\Lambda V^{-1}\vec{x}$

$V^{-1}\frac{d}{dt}\vec{x} = \Lambda V^{-1}\vec{x}$

$\frac{d}{dt}\underbrace{V^{-1}\vec{x}}_{\vec{z}} = \Lambda \underbrace{V^{-1}\vec{x}}_{\vec{z}}$

Define $\vec{z} = V^{-1}\vec{x}$

$\frac{d}{dt}\vec{z} = \Lambda\vec{z}$

vector diagonal

d) Solve the differential equation for $z_i(t)$.

$\Lambda = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 0 & -2 \end{bmatrix}$

$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

Initial Condition

$\vec{z}(0) = V^{-1}\vec{x}(0)$

$= \begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Scalar Diff Eqs. $\rightarrow$ $\frac{dz_1}{dt} = -5z_1 \rightarrow z_1 = z_1(0)e^{-5t}$

$\rightarrow$ $\frac{dz_2}{dt} = -2z_2 \rightarrow z_2 = z_2(0)e^{-2t}$

$z_1(t) = e^{-5t}$

$z_2(t) = 2e^{-2t}$

e) Convert your solutions $z_i(t)$ back into the original variables to find the solution $x_i(t)$.

$$\vec{x} = V \cdot \vec{z} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} e^{-5t} \\ 2e^{-2t} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{-5t} + 2e^{-2t} \\ -e^{-5t} + 4e^{-2t} \end{bmatrix}$$

f) We can solve this equation using a slightly shorter approach by observing that the solutions for $x_i(t)$ will all be of the form

$$x_i(t) = \sum_k c_k e^{\lambda_k t}$$

where λ_k is an eigenvalue of our differential equation relation matrix A.

Since we have observed that the solutions will include $e^{\lambda_i t}$ terms, once we have found the eigenvalues for our differential equation matrix, we can guess the forms of the $x_i(t)$ as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t} \\ \beta_1 e^{\lambda_1 t} + \beta_2 e^{\lambda_2 t} \end{bmatrix}$$

where α_1, α_2, β_1, β_2 are all constants.

Take the derivative to write out

$$\begin{bmatrix} \frac{d}{dt} x_1(t) \\ \frac{d}{dt} x_2(t) \end{bmatrix}.$$

and connect this to the given differential equation.

Solve for $x_i(t)$ from this form of the derivative.

$$\vec{x} = V \vec{z}$$

$$\vec{z} = \begin{bmatrix} k_1 e^{\lambda_1 t} \\ \vdots \\ k_n e^{\lambda_n t} \end{bmatrix}$$

Idea: Guess solution:

$$x_1(t) = \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_1 t}$$

$$x_2(t) = \beta_1 e^{\lambda_1 t} + \beta_2 e^{\lambda_2 t}$$

eigenvalues of A

Let's guess $\vec{x} = \begin{bmatrix} \alpha_1 e^{-5t} + \alpha_2 e^{-2t} \\ \beta_1 e^{-5t} + \beta_2 e^{-2t} \end{bmatrix}$

Then let's take the derivative and set it equal to $A\vec{x}$

$$\frac{d}{dt}\vec{x} = \begin{bmatrix} -5\alpha_1 e^{-5t} - 2\alpha_2 e^{-2t} \\ -5\beta_1 e^{-5t} - 2\beta_2 e^{-2t} \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix}\vec{x} = \begin{bmatrix} -4\alpha_1 e^{-5t} - 4\alpha_2 e^{-2t} + \beta_1 e^{-5t} + \beta_2 e^{-2t} \\ 2\alpha_1 e^{-5t} + 2\alpha_2 e^{-2t} - 3\beta_1 e^{-5t} - 3\beta_2 e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} (-4\alpha_1 + \beta_1) e^{-5t} + (-4\alpha_2 + \beta_2) e^{-2t} \\ (2\alpha_1 - 3\beta_1) e^{-5t} + (2\alpha_2 - 3\beta_2) e^{-2t} \end{bmatrix}$$

Matching coefficients, we see that

same eqs.

$$\left[\begin{matrix} -5\alpha_1 = -4\alpha_1 + \beta_1 \\ -5\beta_1 = 2\alpha_1 - 3\beta_1 \end{matrix}\right. \qquad \left.\begin{matrix} -2\alpha_2 = -4\alpha_2 + \beta_2 \\ -2\beta_2 = 2\alpha_2 - 3\beta_2 \end{matrix}\right]$$

or $\alpha_1 = -\beta_1$ and $2\alpha_2 = \beta_2$

Then plug in the initial condition

$$\vec{x}(0) = \begin{bmatrix} \alpha_1 + \alpha_2 \\ \beta_1 + \beta_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Now we have 4 eqs.

$$\alpha_1 + \alpha_2 = 3 \qquad \beta_1 + \beta_2 = 3$$

Comes from substituting $\alpha_1 = -\beta_1$, $2\alpha_2 = \beta_2$ into $\beta_1 + \beta_2 = 3$

$$\left[-\alpha_1 + 2\alpha_2 = 3 \qquad -\beta_1 + \tfrac{1}{2}\beta_2 = 3\right.$$

$$3\alpha_2 = 6 \qquad \tfrac{3}{2}\beta_2 = 6$$

$$\boxed{\begin{matrix} \alpha_2 = 2 \\ \alpha_1 = 1 \end{matrix}} \qquad \boxed{\begin{matrix} \beta_2 = 4 \\ \beta_1 = -1 \end{matrix}}$$

Final Sol is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{-5t} + 2e^{-2t} \\ -e^{-5t} + 4e^{-2t} \end{bmatrix}$$

Alternate Method of Guess & Check

Let's guess $\vec{x} = \begin{bmatrix} \alpha_1 e^{-5t} + \alpha_2 e^{-2t} \\ \beta_1 e^{-5t} + \beta_2 e^{-2t} \end{bmatrix}$

The initial conditions are $x_1(0) = 3$, $x_2(0) = 3$ and they tell us that $\vec{x}(0) = \begin{bmatrix} \alpha_1 + \alpha_2 \\ \beta_1 + \beta_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

Then

$$\frac{d}{dt}\vec{x} = \begin{bmatrix} -5\alpha_1 e^{-5t} - 2\alpha_2 e^{-2t} \\ -5\beta_1 e^{-5t} - 2\beta_2 e^{-2t} \end{bmatrix} \quad \text{so}$$

$$\frac{d}{dt}\vec{x}(0) = \begin{bmatrix} -5\alpha_1 - 2\alpha_2 \\ -5\beta_1 - 2\beta_2 \end{bmatrix}$$

But we know $\frac{d}{dt}\vec{x} = A\vec{x}$ so $\frac{d}{dt}\vec{x}(0) = A\vec{x}(0)$

$$A\vec{x}(0) = \begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -9 \\ -3 \end{bmatrix}$$

So $\begin{bmatrix} -5\alpha_1 - 2\alpha_2 \\ -5\beta_1 - 2\beta_2 \end{bmatrix} = \begin{bmatrix} -9 \\ -3 \end{bmatrix}$

and $\begin{bmatrix} \alpha_1 + \alpha_2 \\ \beta_1 + \beta_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

$\alpha_1 + \alpha_2 = 3$
$-5\alpha_1 - 2\alpha_2 = -9$
$-3\alpha_1 = -3$
$\alpha_1 = 1, \quad \alpha_2 = 2$

$\beta_1 + \beta_2 = 3$
$-5\beta_1 - 2\beta_2 = -3$
$-3\beta_1 = 3$
$\beta_1 = -1, \quad \beta_2 = 4$

Final Sol: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{-5t} + 2e^{-2t} \\ -e^{-5t} + 4e^{-2t} \end{bmatrix}$