# 1 Inductors: Introduction

So far in the class, we have learnt about capacitors. A capacitor typically consists of parallel metal plates separated by non-conducting material. As charge deposits on the metal plates, we have a resulting electric field and electric potential across the metal plates.

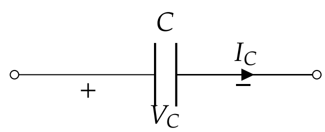



Figure 1: Capacitor element.

This charge-voltage relationship is what we have seen earlier.

$$Q_C \propto V_C \tag{1}$$

The proportionality constant for Equation 1 is a physical property of the capacitor and called its capacitance $C$. Specifically, we have $Q_C = CV_C$. Since current is defined as the rate of flow of charge, we can write

$$I_C = \frac{dQ_C}{dt} = C\frac{dV_C}{dt}. \tag{2}$$

An inductor converts electrical current into magnetic flux. We can construct an inductor by winding a wire into a coil and passing current through it.

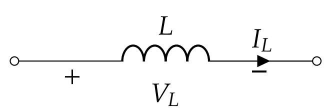



Figure 2: Inductor element.

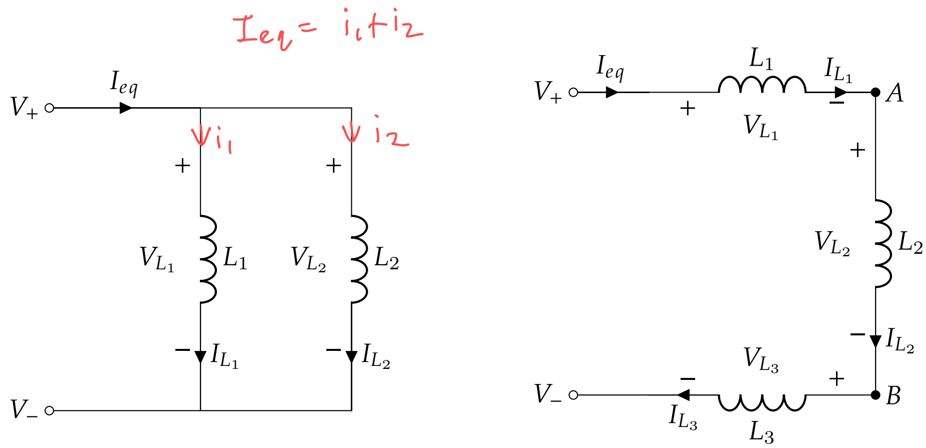
The magnetic flux that develops as a result of the current flowing in a loop is proportional to the loop current.

$$\phi_L \propto I_L \tag{3}$$

Again the proportionality constant for Equation 3 is a physical property of the inductor and called its inductance $L$. Electric potential, or voltage, is related to magnetic flux as

$$V_L = \frac{d\phi_L}{dt} = L\frac{dI_L}{dt}. \tag{4}$$

From the point of view of current-voltage relationships, a capacitor and inductor are *duals* of each other.




(a) Inductors connected in parallel.

(b) Inductors connected in series.

Figure 3: Finding equivalent inductance.

a) For the circuit shown in Figure 3a, find the equivalent inductance across the nodes $V_+$ and $V_-$ for inductors connected in parallel.

Key Idea: In parallel, the voltages are the same, the currents branch off due to KCL

$I_{eq} = I_1 + I_2$ Parallel, $V_1 = V_2 = V_{eq}$ $V = L\frac{di}{dt}$

$\frac{d}{dt} I_{eq} = \frac{dI_1}{dt} + \frac{dI_2}{dt}$ $\frac{di}{dt} = \frac{V}{L}$

$\frac{\cancel{V_{eq}}}{L_{eq}} = \frac{\cancel{V_{eq}}}{L_1} + \frac{\cancel{V_{eq}}}{L_2} \rightarrow \frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2}$ $L_{eq} = \left(\frac{1}{L_1} + \frac{1}{L_2}\right)^{-1} = L_1 \parallel L_2$

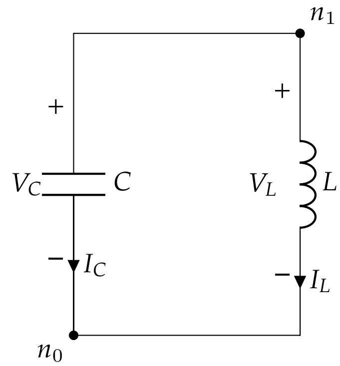
b) For the circuit shown in Figure 3b, find the equivalent inductance across the nodes $V_+$ and $V_-$ for inductors connected in series.

$L_{eq} = L_1 + L_2 + L_3$

Key Idea: In series, the currents are the same, the voltages sum up by KVL

## 2 LC Tank: Oscillations

Consider the following circuit.




This is sometimes called an $LC$ tank and we will look at its response in this problem. Assume at $t = 0$ we have $V_C(0) = V_S = 1$ V and $I_L(0) = 0$. For numerical calculations, use $C = 1uF$, $L = 10mH$.

a) **Write the system of differential equations in terms of state variables $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ that describes this circuit for $t \geq 0$.** Leave the system symbolic in terms of $V_s$, $L$, and $C$.

KCL: $I_C = -I_L$ Want: $\frac{d\vec{x}}{dt} = A\vec{x}$

$\frac{dx_1}{dt} = \frac{dI_L}{dt} = \frac{V_L}{L} = \frac{V_C}{L}$ Know: Capacitor $I_C = C\frac{dV_C}{dt}$

Inductor $V_L = L\frac{dI_L}{dt}$

$\frac{dx_2}{dt} = \frac{dV_C}{dt} = \frac{I_C}{C} = -\frac{I_L}{C}$

KVL: $V_C = V_L$

$\frac{dx_1}{dt} = \frac{x_2}{L}$ $\frac{dx_2}{dt} = -\frac{x_1}{C}$

$A = \begin{bmatrix} 0 & 1/L \\ -1/C & 0 \end{bmatrix}$

b) In later problems, we will use diagonalization to solve for the inductor current $I_L(t)$ and the capacitor voltage $V_C(t)$. The diagonalization approach is more general and applicable to more complex circuits comprised of resistive elements. For this circuit, observe that the capacitor voltage and inductor current in this circuit obey

Simple harmonic oscillator

$\frac{d^2x}{dt} + ax = 0$

$$\frac{d^2}{dt^2}I_L(t) = \frac{-1}{LC}I_L(t) \quad (5)$$

$$\frac{d^2}{dt^2}V_C(t) = \frac{-1}{LC}V_C(t) \quad (6)$$

This expression describes a simple harmonic oscillator.

$\frac{dI_L}{dt} = \frac{V_C}{L} \rightarrow \frac{d^2I_L}{dt^2} = \frac{1}{L}\frac{dV_C}{dt} = \frac{1}{LC}I_C = -\frac{1}{LC}I_L$

**Verify that $V_C(t) = A\cos(\omega t + \theta)$, and $I_L(t) = B\sin(\omega t + \theta)$ is a solution to the system of differential equations originally derived in part (a). Determine the oscillation frequency $\omega$, initial phase $\theta$ and scalar constants $A$ and $B$.**

Want to do: Guess and check $V_c = A\cos(\omega t + \theta)$ and solve for $A, \omega, \theta$.

Hint: Use all information you have including initial cond.

$V_c(0) = 1, \quad I_L(0) = 0$

Plug into $\frac{d^2 V_c}{dt} = -\frac{1}{LC} V_c$

$V_c = A\cos(\omega t + \theta)$

$\frac{d}{dt} V_c = -A\omega \sin(\omega t + \theta)$

$\frac{d^2}{dt^2} V_c = -A\omega^2 \cos(\omega t + \theta)$

$-\cancel{A}\omega^2 \cos \cancel{(\omega t + \theta)} = -\frac{1}{LC} \cancel{A} \cos \cancel{(\omega t + \theta)}$

$-\omega^2 = -\frac{1}{LC} \qquad \omega = \sqrt{\frac{1}{LC}}$

$\omega^2 = \frac{1}{LC}$

$V_c(0) = A\cos(0 \cdot t + \theta)$

$= A\cos\theta = 1$

$\frac{d}{dt} V_c(0) = -A \frac{1}{\cancel{\sqrt{LC}}} \cdot \sin(\theta) = 0$

$= A \sin\theta = 0$

$A = 0$ not possible

$\sin\theta = 0$

$\theta = \underline{0}, \pi$

$\frac{d}{dt} V_c = \frac{I_c}{C} = \frac{-I_L}{C}$

$\frac{d}{dt} V_c(0) = -\frac{I_L(0)}{C}$

$A = 1, \cos\theta = 0$

$V_c = \cos(\omega t)$

$= -\cos(\omega t + \pi)$

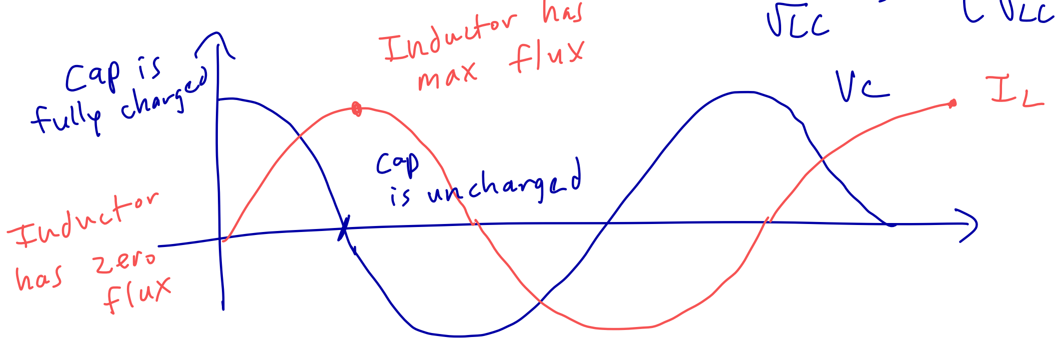
$I_L = -C \frac{dV_c}{dt}$

c) For capacitance $C = 1\text{uF}$ and $L = 10\text{mH}$, **Sketch the capacitor voltage $V_C(t)$ and inductor current $I_L(t)$**. What is happening to the capacitor charge $Q_C$ and inductor flux $\phi_L$.

$V_c = \cos(\omega t) \qquad I_L = -\omega C(-\sin \omega t)$

$= \cos\left(\frac{1}{\sqrt{LC}} t\right) \qquad = \omega C \sin \omega t$

$= \frac{C}{\sqrt{LC}} \sin\left(\frac{1}{\sqrt{LC}} t\right)$




$Q = CV_c$

$\phi = LI \qquad I = 0 \rightarrow \phi = 0$

$V_c = V_S \cos\left(\frac{1}{\sqrt{LC}} t\right)$



d) The energy stored in the capacitor is given by $E_C = \frac{1}{2}CV_C^2$ and the energy stored in the inductor is given by $E_L = \frac{1}{2}LI_L^2$. **Evaluate how the total energy in the circuit is changing with time.**

$V_c = \cos\left(\frac{1}{\sqrt{LC}} t\right)$ $\quad I_L = \frac{C}{\sqrt{LC}} \sin\left(\frac{1}{\sqrt{LC}} t\right)$

$E_c = \frac{1}{2} C \cdot \cos^2\left(\frac{1}{\sqrt{LC}} t\right)$ $\quad E_L = \frac{1}{2} \cancel{L} \cdot \frac{C^2}{\cancel{L}C} \sin^2\left(\frac{1}{\sqrt{LC}} t\right)$

$= \frac{1}{2} C \sin^2\left(\frac{1}{\sqrt{LC}} t\right)$

$E_{total} = E_c + E_L$

$= \frac{1}{2} C \cos^2\left(\frac{1}{\sqrt{LC}} t\right) + \frac{1}{2} C \sin^2\left(\frac{1}{\sqrt{LC}} t\right)$

$= \frac{1}{2} C V_s^2$

capacitance

e) We will now use diagonalization to get to the same solution that we have analyzed so far. **Write the system of equations in vector/matrix form with the vector state variable** $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.
This should be in the form $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$ with a $2 \times 2$ matrix $A$.
**Find the initial conditions** $\vec{x}(0)$.

Repeat of part (a):

$A = \begin{bmatrix} 0 & 1/L \\ -1/C & 0 \end{bmatrix}$ $\quad \frac{d}{dt}\vec{x} = \begin{bmatrix} 0 & 1/L \\ -1/C & 0 \end{bmatrix}$

$\vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

f) **Find the eigenvalues of the $A$ matrix symbolically.**

$A = \begin{bmatrix} 0 & 1/L \\ -1/C & 0 \end{bmatrix}$ $\quad \det(A - \lambda I) = \lambda^2 + \frac{1}{LC}$

$\lambda = \pm j \sqrt{\frac{1}{LC}}$

$\uparrow$

complex eigenvalues

g) Recall from our previous discussion that solutions for $x_i(t)$ will all be of the form

$$x_i(t) = \sum_k c_k e^{\lambda_k t}$$

where $\lambda_k$ is an eigenvalue of our differential equation relation matrix $A$. Thus, we make the following guess for $\vec{x}(t)$:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ c_3 e^{\lambda_1 t} + c_4 e^{\lambda_2 t} \end{bmatrix}$$

where $c_1,\ c_2,\ c_3,\ c_4$ are all constants.

**Evaluate $\vec{x}(t)$ and $\frac{d\vec{x}}{dt}(t)$ at time $t = 0$ in order to obtain four equations in four unknowns.**

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & 0 \end{bmatrix} \vec{x} \qquad \vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad x_1 = I_L \quad x_2 = V_C$$

$$\frac{d\vec{x}}{dt}(0) = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & 0 \end{bmatrix} \vec{x}(0) = \begin{bmatrix} 0 & 1/L \\ -1/C & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/L \\ 0 \end{bmatrix}$$

h) **Solve those equations for $c_1$, $c_2$, $c_3$, $c_4$ and plug them into your guess for $\vec{x}(t)$.** What do you notice about the solutions? Are they complex functions?

$$\vec{x}(t) = \begin{bmatrix} c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ c_3 e^{\lambda_1 t} + c_4 e^{\lambda_2 t} \end{bmatrix} \qquad \vec{x}(0) = \begin{bmatrix} c_1 + c_2 \\ c_3 + c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\frac{d\vec{x}}{dt}(t) = \begin{bmatrix} \lambda_1 c_1 e^{\lambda_1 t} + \lambda_2 c_2 e^{\lambda_2 t} \\ \lambda_1 c_3 e^{\lambda_1 t} + \lambda_2 c_4 e^{\lambda_2 t} \end{bmatrix} \qquad \frac{d}{dt}\vec{x}(0) = \begin{bmatrix} \lambda_1 c_1 + \lambda_2 c_2 \\ \lambda_1 c_3 + \lambda_2 c_4 \end{bmatrix} = \begin{bmatrix} 1/L \\ 0 \end{bmatrix}$$

$$c_1 + c_2 = 0$$
$$\lambda_1 c_1 + \lambda_2 c_2 = \frac{1}{L}$$
$$-\lambda_2 c_1 - \lambda_2 c_2 = -\lambda_2 \cdot 0$$
$$\lambda_1 c_1 + \lambda_2 c_2 = \frac{1}{L}$$
$$(\lambda_1 - \lambda_2) c_1 = \frac{1}{L}$$
$$\text{Let } \lambda_1 = j\sqrt{\frac{1}{LC}}, \quad \lambda_2 = -j\sqrt{\frac{1}{LC}}$$
$$2j \frac{1}{\sqrt{LC}} \cdot c_1 = \frac{1}{L}$$
$$c_1 = \frac{1}{2j}\sqrt{\frac{C}{L}}$$
$$c_2 = -c_1 = -\frac{1}{2j}\sqrt{\frac{C}{L}}$$

$$c_3 + c_4 = 1$$
$$\lambda_1 c_3 + \lambda_2 c_4 = 0$$
$$-\lambda_2 c_3 - \lambda_2 c_4 = -\lambda_2$$
$$\lambda_1 c_3 + \lambda_2 c_4 = 0$$
$$(\lambda_1 - \lambda_2) c_3 = -\lambda_2$$
$$2j \frac{1}{\sqrt{LC}} c_3 = j \frac{1}{\sqrt{LC}}$$
$$c_3 = \frac{1}{2}$$
$$c_4 = 1 - c_3 = \frac{1}{2}$$

Euler's Formula

$$e^{j\omega t} + e^{-j\omega t} = 2\cos\omega t$$



$$V_C = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) = \cos(\omega t)$$

$$I_L = \frac{1}{2j}\sqrt{\frac{C}{L}}\left(e^{j\omega t} - e^{-j\omega t}\right) = \sqrt{\frac{C}{L}}(\sin \omega t)$$

$$e^{j\omega t} - e^{-j\omega t} = 2j\sin\omega t$$