1 State Space Models

There are many kinds of **dynamical systems** we might want to study or control. Some examples are an airplane's flight, the air inside a building, or network traffic on the internet. We can develop controllers for these systems to regulate particular quantities that we care about, like an autopilot to level an airplane's flight, a thermostat to keep a building at a comfortable temperature, or internet congestion control to manage data rates. Other dynamical systems and controllers can be found in nature, like the biochemical systems that regulate conditions inside a living cell.

When we want to study or control a dynamical system, our first step is usually to write out equations that describe its physics. These equations are called a **model**, and they predict what a system will do over time. We will study systems that change continuously in time like electrical circuits, and systems that evolve in discrete time steps, like the yearly number of professors in EECS.

State variables are a set of variables that fully represent the state of a dynamical system at a given time, like capacitor voltages and inductor currents in electrical circuits. In a mechanical system, they could be the positions and velocities of masses. The state variables can be written together in a **state vector** $\vec{x}(t) \in \mathbb{R}^n$ where n is the number of state variables that describe the system.

2 Continuous Systems

For a **continuous-time system**, the dynamics can be described by n first-order differential equations:

$$\frac{d}{dt}\vec{x}(t) = f(\vec{x}(t))$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a function of the state vector that returns the time derivative of the state vector (which is an n-dimensional vector containing the time derivative of each state variable).

A system with m input signals can be described as:

$$\frac{d}{dt}\vec{x}(t) = f(\vec{x}(t), \vec{u}(t))$$

where $\vec{u}(t) \in \mathbb{R}^m$ is a **control input** with which we can vary to influence the system.

We can expand out this vector dynamics equation:

$$\begin{bmatrix} \frac{d}{dt}x_1(t) \\ \vdots \\ \frac{d}{dt}x_n(t) \end{bmatrix} = \begin{bmatrix} f_1(\vec{x}(t), \vec{u}(t)) \\ \vdots \\ f_n(\vec{x}(t), \vec{u}(t)) \end{bmatrix},$$

where $f_i(\vec{x}, \vec{u}(t))$ returns the time derivative of the ith state variable. A continuous-time system is **linear** if it can be expressed in the form $\frac{d}{dt}\vec{x}(t) = \mathbf{A}\vec{x}(t) + \mathbf{B}\vec{u}(t)$:

3 Discrete Time Systems

For a **discrete-time system**, the dynamics can be described by n difference equations:

$$\vec{x}[t+1] = f(\vec{x}[t], \vec{u}[t]),$$

where $\vec{x}[t+1]$ is the new state vector at the next time step.

As in the continuous case, a linear discrete-time system's dynamics can be written as:

$$\vec{x}[t+1] = \mathbf{A}\vec{x}[t] + \mathbf{B}\vec{u}[t]$$

4 Dirty Dishes

I am a trip planner who lodges travellers at Bob's Bed and Breakfast. At the beginning of each day, Bob will do half of the dirty dishes in the sink. During the day, each of his guests will use 4 pounds of dishes minus an eighth pound of dishes for each pound of dishes already in the sink at the beginning of the day (as Bob's kitchen gets too messy).

a) Define a state vector for Bob's kitchen sink system. Also, what are the inputs? Lastly, write out the state space model using your state vector and inputs.

x: pounds of dirty dishes at the end of each day

u: number of guests   Discrete Time

$$x[t+1] = \left(4 - \tfrac{1}{8}x[t]\right)u[t] + \tfrac{1}{2}x[t]$$

b) Explain why Bob's kitchen is not a linear system.

$x(t)$ is multiplied by $u(t)$, not a linear system

$t=0$

$x[0]=4$

$u[0]=4$

$u[1]=5$

c) On Wednesday morning (before Bob gets up), there are 4 pounds of dishes in the sink. On Wednesday, Bob has 4 guests, and on Thursday, he has 5 guests. How many pounds of dishes are in the sink after Thursday? at the beginning of Friday or $t=2$

$$x[1] = \left(4 - \tfrac{1}{8}x[0]\right)u[0] + \tfrac{1}{2}x[0]$$
$$= \left(4 - \tfrac{1}{2}\right)\cdot 4 + 2 = \tfrac{7}{2}\cdot 4 + 2 = 16$$
$$x[2] = \left(4 - \tfrac{1}{8}x[1]\right)u[1] + \tfrac{1}{2}x[1]$$
$$= \left(4 - \tfrac{1}{8}\cdot 16\right)\cdot 5 + \tfrac{1}{2}\cdot 16$$
$$= (4-2)\cdot 5 + 8 = 18$$

d) I am a very eccentric trip planner and I want Bob to have exactly 12 pounds of dishes in his sink. He has 24 pounds of dishes in his sink. How many guests should I lodge at Bob's Bed and Breakfast today? How many guests should I lodge tomorrow?

$x[0] = 24$ want $x[t] = 12$ for $t > 0$

$$x[1] = (4-3)\,u[t] + 12$$
$$= u[t] + 12, \quad u[0] = 0$$
$$x[2] = \left(4 - \tfrac{1}{8}\cdot 12\right)u[1] + \tfrac{1}{2}\cdot 12$$
$$12 = \left(4 - \tfrac{3}{2}\right)u[1] + 6$$

$$6 = \tfrac{5}{2}u[1] \rightarrow u[1] = \tfrac{12}{5} = 2.4$$

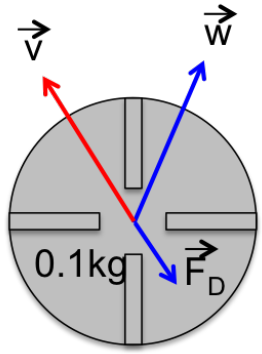
5 Remote Control Hovercraft

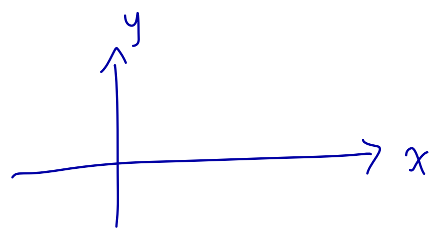
Taejin has a toy hovercraft that he can drive around on the ground. It weighs $0.1\,\mathrm{kg}$. His remote control has two levers: one sets the thrust in the x-direction, w_x, measured in Newtons, and the other sets the thrust in the y-direction, w_y, measured in Newtons. The hovercraft experiences a drag force:

$$\vec{F} = -D\vec{v},$$

where $\vec{F}$ is the drag force vector in Newtons, $\vec{v}$ is the hovercraft velocity vector in $\frac{\mathrm{m}}{\mathrm{s}}$, and D is the coefficient $0.05\,\frac{\mathrm{N\,s}}{\mathrm{m}}$.

Forces: Drag force
Thrust





a) If we are interested in both the position and velocity of our hovercraft, define the appropriate state variables and inputs for the hovercraft system.

4 state variables

x: position in x $\quad v_x$: velocity in x

y: position in y $\quad v_y$: velocity in y

Inputs

$w_x,\ w_y$

$\vec{u} = \begin{bmatrix} w_x \\ w_y \end{bmatrix}$

b) Write out the state space model using the state variables and inputs you identified.

Hint: Newton's Second Law states that $F_{net} = ma$. Also, remember that $a(t) = v'(t)$ and $v(t) = x'(t)$.

$a = \frac{d}{dt} v(t), \quad v(t) = \frac{d}{dt} x(t)$

$F_{net} = -D\vec{v} + \vec{u} = m\vec{a}$

$F_{net,x} = -D v_x + w_x = m a_x \rightarrow a_x = -\frac{D}{m} v_x + \frac{w_x}{m}$

$\frac{d}{dt} v_x$

$\frac{d}{dt} v_y = -\frac{D}{m} v_y + \frac{w_y}{m}$

$\frac{d}{dt} x = v_x$

$\frac{d}{dt} y = v_y$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ -\frac{D}{m} v_x + \frac{w_x}{m} \\ -\frac{D}{m} v_y + \frac{w_y}{m} \end{bmatrix}$$

$4 \times 2 \cdot 2 \times 1$

c) Is this system linear? If it is, write it in the form $\frac{d}{dt}\vec{x} = \mathbf{A}\vec{x} + \mathbf{B}\vec{u}$. If it isn't, explain why not.

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ -\frac{D}{m} v_x + \frac{w_x}{m} \\ -\frac{D}{m} v_y + \frac{w_y}{m} \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{D}{m} & 0 \\ 0 & 0 & 0 & -\frac{D}{m} \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m} & 0 \\ 0 & \frac{1}{m} \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} w_x \\ w_y \end{bmatrix}$$

d) Taejin places the hovercraft at $(1, 0)$. At $t = 0$, Ramsey kicks the hovercraft, so that it moves at $2\,\frac{\text{m}}{\text{s}}$ in the x direction, and Taejin doesn't touch the remote control. What will the position and velocity of the hovercraft be at $t = 10$?

Hint: Try to solve for the velocity first, then use the fact that $x'(t) = v(t)$ to solve for position.

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{D}{m} & 0 \\ 0 & 0 & 0 & -\frac{D}{m} \end{bmatrix}\begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m} & 0 \\ 0 & \frac{1}{m} \end{bmatrix}\begin{bmatrix} w_x \\ w_y \end{bmatrix}$$

$$\frac{d}{dt} v_x = -\frac{D}{m} v_x \qquad v_y = y = 0 \text{ for all } t$$

$$\frac{d}{dt} x = v_x$$

$$\downarrow$$

$$x(t) = \int_0^t v_x(t)\,dt$$