1 Conditions for Equilibria

Continuous-Time Systems

Let us take a closer look at the conditions for a linear system represented by the differential equation

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t) \tag{1}$$

From the get-go we see that $(\vec{x}^*, \vec{u}^*) = (\vec{0}, \vec{0})$ must be an equilibrium point. This is since the system is at rest. Now if we put in a constant input $\vec{u}^*$ then to solve for equilibria, we get the following system of equations

$$A\vec{x} + B\vec{u}^* = \vec{0} \tag{2}$$

To solve for the states $\vec{x}$ in which the system would be in equilibrium, our analysis boils down to whether the square matrix A is invertible[1].

a) If A is invertible, then there is a unique equilibrium point $\vec{x}^* = -A^{-1}B\vec{u}^*$.

b) If A is non-invertible, depending on the range of A, we have two scenarios.

- If $B\vec{u} \in \text{Col}(A)$ then we will have infinitely many equilibrium points.
- If $B\vec{u} \notin \text{Col}(A)$ then the system has no solution and we will have no equilibrium points.

Discrete-Time Systems

Now let's take a look at the discrete-time system

$$\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t) \tag{3}$$

Again we see that $(\vec{0}, \vec{0})$ is an equilibrium point but notice that the conditions for equilibria are different for discrete-time systems. A system is in equilibrium if it is not changing. In otherwords, this means that $\vec{x}^*(t+1]) = \vec{x}^*(t)$ therefore, for a constant input $\vec{u}^*$ we get the following system of equations

$$\vec{x} = A\vec{x} + B\vec{u}^* \implies (I - A)\vec{x} = B\vec{u}^* \tag{4}$$

The conditions for equilibria now depend on the matrix $I - A$ being invertible instead of the matrix A.

[1]This should be review from 16A/54, but we restate it here since it isn't quite obvious when A is singular or non-invertible. Normally a singular matrix has infinite solutions but take the system $A\vec{x} = \vec{b}$ with $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This leads to a contradiction that $x_1 = 0 \neq 1$.

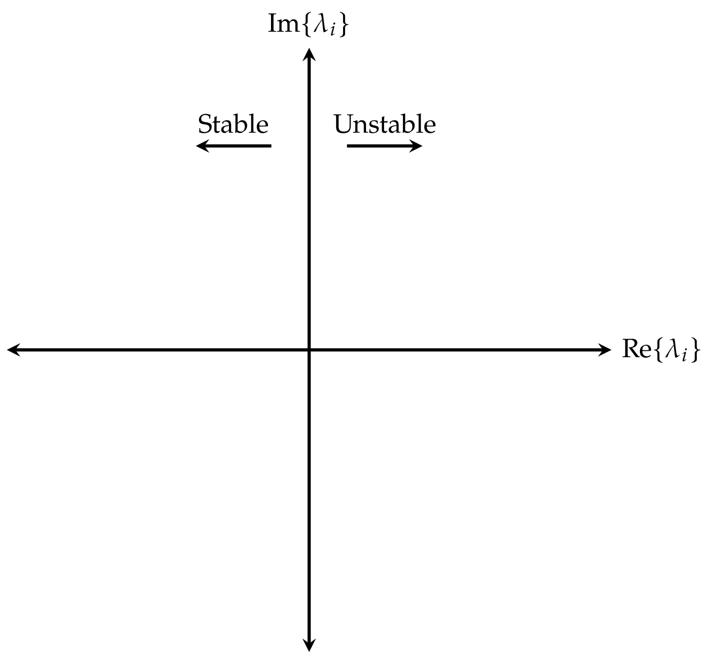
2 Stability

Continuous time systems

A continuous time system is of the form:

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}(t) = A\vec{x}(t) + B\vec{u}(t)$$

This system is stable if $\mathrm{Re}\{\lambda_i\} < 0$ for all λ_i , where λ_i's are the eigenvalues of A. If we plot all λ_i for A on the complex plane, if all λ_i lie to the left of $\mathrm{Re}\{\lambda_i\} = 0$, then the system is stable.



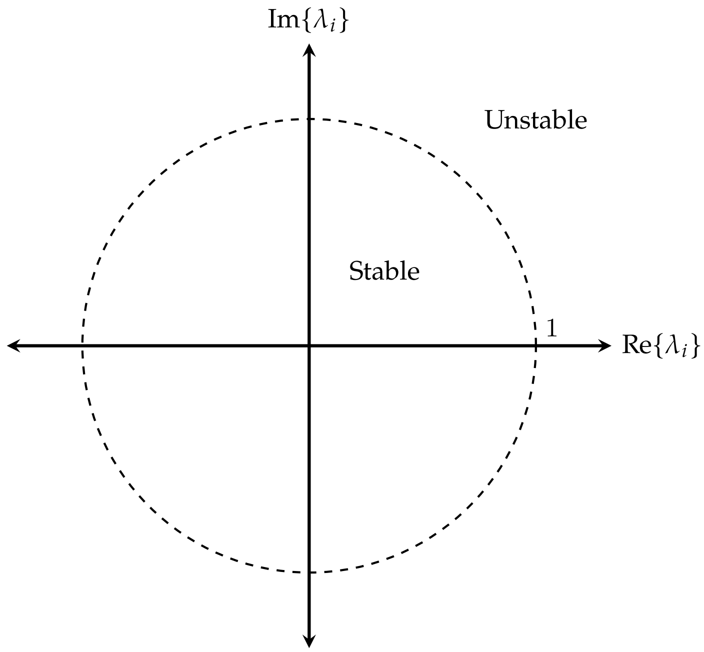
If $\mathrm{Re}\{\lambda_i\} \geq 0$, the system is unstable in the context of BIBO stability.

Discrete time systems

A discrete time system is of the form:

$$\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t)$$

This system is stable if $|\lambda_i| < 1$ for all λ_i, where λ_i's are the eigenvalues of A. If we plot all λ_i for A on the complex plane, if all λ_i lie within (not on) the unit circle, then the system is stable.



If $|\lambda| \geq 1$, we say the system is unstable in the context of Bounded-Input Bounded-Output (BIBO) stability.

3 Jacobian Warm-Up

$\frac{d}{du} \ln(u) = \frac{du}{u}$

Consider the following function $f : \mathbb{R}^2 \mapsto \mathbb{R}^3$

$$f(x_1, x_2) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \\ f_3(x_1, x_2) \end{bmatrix} = \begin{bmatrix} x_1^2 - e^{x_2^2} \\ x_1^2 + \sin(x_1)x_2^2 \\ \log(1 + x_1^2) \end{bmatrix}$$

↑ natural log or ln

Calculate its Jacobian.

Given p functions and n variables, the Jacobian matrix will be of size $p \times n$

Here: 3 functions, 2 variables, $J \in \mathbb{R}^{3 \times 2}$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \end{bmatrix}$$

↑ partials of x_1 ↑ partials of x_2

$$\nabla_{\vec{x}} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}$$

$$J = \begin{bmatrix} 2x_1 & -2x_2 e^{x_2^2} \\ 2x_1 + x_2^2 \cos x_1 & \sin(x_1) 2x_2 \\ \frac{2x_1}{1+x_1^2} & 0 \end{bmatrix}$$

$$\frac{\partial f_1}{\partial x_1} = 2x_1 \qquad \frac{\partial f_1}{\partial x_2} = -e^{x_2^2} \cdot 2x_2$$

$$\frac{\partial f_2}{\partial x_1} = 2x_1 + x_2^2 \cos x_1 \qquad \frac{\partial f_2}{\partial x_2} = \sin(x_1)\, 2x_2$$

$$\frac{\partial f_3}{\partial x_1} = \frac{2x_1}{1 + x_1^2} \qquad \frac{\partial f_3}{\partial x_2} = \frac{0}{1+x_1^2} = 0$$

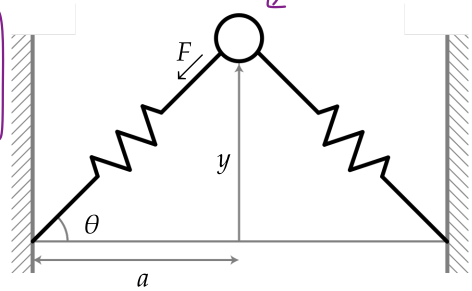
4 Linearization

Ex:

Consider a mass attached to two springs:

$$f = \begin{bmatrix} 4 - x_2^2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_2 = \pm 2$$



General Advice: For an n^{th} order diff-eq pick $x_1 = y$, $x_2 = \frac{dy}{dt}$, ..., $x_n = \frac{d^{n-1}y}{dt^{n-1}}$

$k > 0 \quad m > 0$

We assume that each spring is linear with spring constant k and resting length X_0. We want to build a state space model that describes how the displacement y of the mass from the spring base evolves. The differential equation modeling this system is $\frac{d^2y}{dt^2} = -\frac{2k}{m}\left(y - X_0\frac{y}{\sqrt{y^2+a^2}}\right)$.

a) Write this model in state space form $\dot{x} = f(x)$.

$x_1 = y, \quad x_2 = \frac{dy}{dt}$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{2k}{m}\left(x_1 - X_0\frac{x_1}{\sqrt{x_1^2+a^2}}\right) \end{bmatrix}$$

$$\frac{dx_1}{dt} = \frac{dy}{dt} = x_2$$

$$\frac{dx_2}{dt} = \frac{d^2y}{dt^2} = -\frac{2k}{m}\left(y - X_0\frac{y}{\sqrt{y^2+a^2}}\right) = -\frac{2k}{m}\left(x_1 - X_0\frac{x_1}{\sqrt{x_1^2+a^2}}\right)$$

b) Find the equilibrium of the state-space model. You can assume $X_0 < a$.

Find $\vec{x}^*$, s.t. $f(\vec{x}^*) = 0 \quad \frac{d}{dt}\vec{x} = \vec{0}$

$$\begin{bmatrix} x_2 \\ -\frac{2k}{m}\left(x_1 - X_0\frac{x_1}{\sqrt{x_1^2+a^2}}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

First row: $x_2 = 0$

eq point: $\vec{x}^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Second Row: $-\frac{2k}{m}\left(x_1 - X_0\frac{x_1}{\sqrt{x_1^2+a^2}}\right) = 0$

$$x_1\left(1 - \frac{X_0}{\sqrt{x_1^2+a^2}}\right) = 0$$

Case 1: $x_1 = 0$

Case 2: $1 = \frac{X_0}{\sqrt{x_1^2+a^2}} \rightarrow x_1^2 + a^2 = X_0^2 \quad x_1^2 = X_0^2 - a^2 < 0$

x_1 is imaginary

Contradiction!

$$\sqrt{x_1^2+a^2} = (x_1^2+a^2)^{1/2}$$

c) Linearize your model about the equilibrium.

no input

Find Jacobian $J_{\vec{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$

$$f = \begin{bmatrix} x_2 \\ -\frac{2k}{m}\left(x_1 - \chi_0 \frac{x_1}{\sqrt{x_1^2+a^2}}\right) \end{bmatrix}$$

$\frac{\partial f_1}{\partial x_1} = 0$

$\frac{\partial f_1}{\partial x_2} = 1$

$\frac{\partial f_2}{\partial x_2} = 0$

$$\frac{\partial f_2}{\partial x_1} = -\frac{2k}{m}\left(1 - \chi_0 \frac{\sqrt{x_1^2+a^2}\cdot 1 - x_1 \cdot \frac{1}{2}(x_1^2+a^2)^{-1/2}\cdot(2x_1)}{(x_1^2+a^2)}\right)$$

0

Evaluate at $x_1^* = x_2^* = 0$

$$\frac{\partial f_2}{\partial x_1}(0,0) = -\frac{2k}{m}\left(1 - \frac{\chi_0}{a}\right)$$

d) Compute the eigenvalues of your linearized model. Is this equilibrium stable?

Write out Linearized System

$$\frac{d}{dt}\vec{x}_e = \begin{bmatrix} 0 & 1 \\ -\frac{2k}{m}\left(1 - \frac{x_0}{a}\right) & 0 \end{bmatrix} \vec{x}_e$$

$J_{\vec{x}}$

5 Stability in discrete time system

Determine which values of α and β will make the following discrete-time state space models stable. Assume, α and β are real numbers and $b \neq 0$.

a)

$$x(t+1) = \alpha x(t) + bu(t)$$

b)

$$\vec{x}(t+1) = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \vec{x}(t) + b\vec{u}(t)$$

c)

$$\vec{x}(t+1) = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \vec{x}(t) + b\vec{u}(t)$$