1 Scalar feedback control

$x(t+1) = \lambda x(t)$

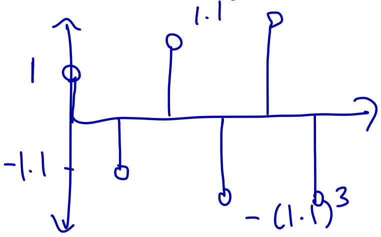
$x(0) = 1$

Suppose that x has the following discrete-time dynamics:

$$x(t+1) = \lambda x(t) + bu(t), \quad x(0) = x_0 \tag{1}$$

a) Assuming that $x_0 = 1$ and $u = 0$, sketch $x(t)$ for a few time steps for $\lambda \in \{-1.1, -1, -0.5, 0.5, 1, 1.1\}$.

$\lambda = -1.1$



for all t

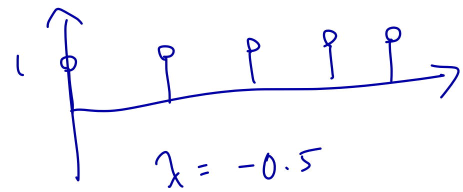
$x(0) = 1$

$x(1) = -1.1$

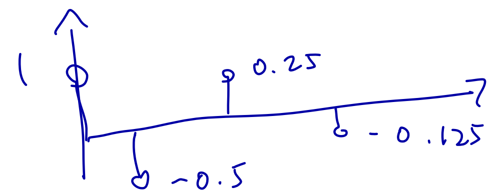
$x(2) = (1.1)^2$

$x(3) = -1.1^3$

$\lambda = 1$ $\quad x(t+1) = x(t)$



$\lambda = -0.5$



$x(0) = 1$

$x(1) = -0.5$

$x(2) = 0.25$

$x(3) = -0.125$

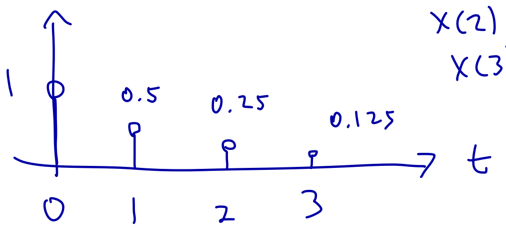
$\lambda = 0.5$

$x(0) = 1$

$x(1) = 0.5$

$x(2) = 0.25$

$x(3) = 0.125$



DT Observations:

1. Negative eigenvalues oscillate
2. $|\lambda| < 1$, x converges to 0
3. $|\lambda| > 1$, x diverges

b) What values of λ will result in convergence of x to its equilibrium? A scalar system having such a λ is called *stable*.

DT System: If $|\lambda| < 1$, then x converges to 0.

CT Observations:

1. Imaginary eigenvalues oscillate
2. $Re(\lambda) < 0$, x converges to 0
3. $Re(\lambda) > 0$, x diverges

c) If $u(t) = u_0$ and the system is stable, what does x converge to? Sketch stable trajectories of x for $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$.

$x(t+1) = \lambda x(t) + bu_0$

$x^* = \lambda x^* + bu_0$

$x^*(1-\lambda) = bu_0$

$x^* = \dfrac{bu_0}{1-\lambda}$

Suppose $x(t)$ converges to x^*

converges if $x(t+1) = x(t)$

This all assumes $|\lambda| < 1$

but as $\lambda \to 1$ $\quad x^* \to \infty$

as $\lambda \to -1$ $\quad x^* \to \dfrac{bu_0}{2}$

Suppose $\lambda = 1$, $u(t) = u_0$ $\quad x(t+1) = x(t) + bu_0$

$x(0) = 1 \quad x(1) = 1 + bu_0 \quad x(2) = x(1) + bu_0 = 1 + 2bu_0$

$x(3) = 1 + 3bu_0 \leftarrow$ grows to ∞ as $t \to \infty$ unstable!

d) If $x(t+1) = \lambda x(t) + bu(t)$ is unstable, describe feedback laws $u(t) = kx(t)$ that stabilize the equilibrium $x = 0$.

Find k that stabilize the system

$$x(t+1) = \lambda x(t) + bk\, x(t)$$
$$= (\lambda + bk)\, x(t)$$

Closed-loop system is stable when $|\lambda + bk| < 1$

$$-1 < \lambda + bk < 1$$
$$-1 - \lambda < bk < 1 - \lambda$$

Assume $b > 0$ $\quad \frac{-1-\lambda}{b} < k < \frac{1-\lambda}{b}$

if $b < 0$

$k > \frac{1-\lambda}{b}$, $k < \frac{-1-\lambda}{b}$

e) Now, consider the continuous time system

$$\frac{d}{dt}x(t) = \lambda x(t) + bu(t) \tag{2}$$

Consider the case where this system is unstable ($\lambda \geq 0$). Design a feedback law $u(t) = kx(t)$ which stabilizes the equilibrium $x = 0$. You can assume that $b > 0$.

$$\text{Re}(\lambda + bk) < 0$$

2 Eigenvalues Placement in Discrete Time

Consider the following linear discrete time system

noise or disturbances ideally are close to 0

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}}_{A} \vec{x}(t) + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u(t) + w(t) \tag{3}$$

a) Is this system controllable?

$$C = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Rank $C = 2$, C is full rank so system is controllable

b) Is the linear discrete time system stable?

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 2 & -1-\lambda \end{vmatrix} = (-\lambda)(-1-\lambda) - 2$$
$$= \lambda(\lambda+1) - 2$$
$$= \lambda^2 + \lambda - 2$$
$$= (\lambda+2)(\lambda-1)$$

$\lambda = 1, -2$ since $|\lambda| \geq 1$, system is unstable

what if $\lambda = \frac{1}{2}, -2$, since $|-2| = 2 \geq 1$, system is unstable

c) Derive a state space representation of the resulting closed loop system using state feedback of the form $u(t) = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \vec{x}(t)$

$$\vec{x}(t+1) = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}(t) + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{2\times1} \underbrace{\begin{bmatrix} k_1 & k_2 \end{bmatrix}}_{1\times2} \vec{x}(t)$$

$$= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix} \vec{x}(t)$$

$$A_{cl} = \begin{bmatrix} k_1 & 1+k_2 \\ 2 & -1 \end{bmatrix}$$

$$\vec{x}(t+1) = A_{cl} \vec{x}(t)$$

d) Find the appropriate state feedback constants, k_1, k_2 in order the state space representation of the resulting closed loop system to place the eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$

Want our characteristic polynomial to be $(\lambda + \frac{1}{2})(\lambda - \frac{1}{2}) = \lambda^2 - \frac{1}{4}$

Goal: find k_1, k_2 s.t. $\det(A_{cl} - \lambda I) = \lambda^2 + 0 \cdot \lambda - \frac{1}{4}$

$$\det(A_{cl} - \lambda I) = \begin{vmatrix} k_1 - \lambda & 1+k_2 \\ 2 & -1-\lambda \end{vmatrix} = (k_1 - \lambda)(-1-\lambda) - 2 - 2k_2$$
$$= (\lambda - k_1)(\lambda+1) - 2 - 2k_2$$
$$= \lambda^2 + (1-k_1)\lambda - k_1 - 2 - 2k_2$$

Want coefficients to match

$1 - k_1 = 0 \rightarrow k_1 = 1$

$-k_1 - 2 - 2k_2 = -\frac{1}{4} \rightarrow -3 - 2k_2 = -\frac{1}{4} \rightarrow 2k_2 = -3 + \frac{1}{4} = -\frac{11}{4}$

$k_2 = -\frac{11}{8}$

e) Suppose that instead of $\begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$ in (3), we had $\begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$ as the way that the discrete-time control acted on the system. Is this system controllable from $u(t)$?

f) For the part above, suppose we used $[k_1, k_2]$ to try and control the system. What would the eigenvalues be? Can you move all the eigenvalues to where you want? Give an intuitive explanation of what is going on.

uncontrollable, $\lambda_1 = -\frac{1}{2}$, $\lambda_2 = \frac{1}{2}$

This isn't guaranteed to be possible