EECS 16B Designing Information Devices and Systems II
Fall 2020 UC Berkeley

Note 11

1 Introduction

In the previous note, we discussed different types of state-space systems and ended on the note that linear systems are desirable. Linear models are advantageous because they have predictable solutions and its control can be studied using Linear Algebra.

However, most systems in the real world are nonlinear due to disturbances, noise, or internal resistive forces. The methods applicable to nonlinear models are limited and will often not have a closed form solution. Therefore it is common practice to approximate a nonlinear model with a linear one that is valid around a desired operating point. This type of analysis is called **Linearization** and will be the focus of this note.

2 Taylor Expansion

The key to linearization is understanding the multivariate Taylor Expansion of a function f. Recall from Calculus that we can expand a scalar function $f(x)$ around a point x_0 through its Taylor Expansion as

$$f(x) = f(x_0) + f'(x_0) \cdot (x - x_0) + \frac{1}{2} f''(x_0) \cdot (x - x_0)^2 + \dots \tag{1}$$

To create a first-order approximation f_ℓ to the function f, we keep the first two terms and truncate the rest:

$$f_\ell(x) = f(x_0) + f'(x_0) \cdot (x - x_0) \tag{2}$$

The function $f_\ell(x)$ will approximate f well for values of x close to x_0. This is because the higher order terms in the Taylor Series of f, $\frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$ are close to zero.

Now that we've seen how Taylor series work in the scalar case, let's try to extend it to the multivariate case. If we have 2 multivariate functions f_1 and f_2 with respect to the variables x and y, we can expand each one around the point x_0, y_0 to see that

$$f_1(x,y) = f_1(x_0,y_0) + \frac{\partial f_1}{\partial x}(x_0,y_0) \cdot (x - x_0) + \frac{\partial f_1}{\partial y}(x_0,y_0) \cdot (y - x_0) + \dots \tag{3}$$

$$f_2(x,y) = f_2(x_0,y_0) + \frac{\partial f_2}{\partial x}(x_0,y_0) \cdot (x - x_0) + \frac{\partial f_2}{\partial y}((x_0,y_0) \cdot (y - y_0) + \dots \tag{4}$$

The first-order approximation of f can be expressed in matrix-vector form as follows,

$$\begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} = \begin{bmatrix} f_1(x_0,y_0) \\ f_2(x_0,y_0) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial x}(x_0,y_0) & \frac{\partial f_1}{\partial y}(x_0,y_0) \\ \frac{\partial f_2}{\partial x}(x_0,y_0) & \frac{\partial f_2}{\partial y}(x_0,y_0) \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} \tag{5}$$

$$= \begin{bmatrix} f_1(x_0,y_0) \\ f_2(x_0,y_0) \end{bmatrix} + J(x_0,y_0) \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} \tag{6}$$

3 Jacobian Matrices

In equation (6), the matrix J contains all of the partial derivatives of the functions f_1 and f_2 and is called the **Jacobian**. Extending the Jacobian to n functions $f_1, \ldots, f_n$, in terms of n variables, $x_1, \ldots x_n$, we see that

$$J_{\vec{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}.$$

This lets us write out the f_ℓ about $\vec{x}^*$ as a matrix-vector equation using the Jacobian.

$$f_\ell(\vec{x}) = f(\vec{x}^*) + J_{\vec{x}}(\vec{x}^*) \cdot (\vec{x} - \vec{x}^*) = f(\vec{x}^*) + J_{\vec{x}}(\vec{x}^*) \cdot \vec{x}_\ell(t) \tag{7}$$

For ease of notation, we define $\vec{x}_\ell = \vec{x} - \vec{x}^*$ as the linearized state-vector. Intuitively this represents the distance of the state $\vec{x}$ from the operating point $\vec{x}^*$.

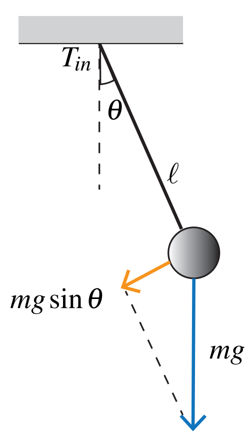
$$\boxed{\vec{x}_\ell(t) \triangleq \vec{x}(t) - \vec{x}^*} \tag{8}$$

4 Equilibrium Points

Which points should we linearize the system around? Are some points better than others? In this section, we discuss the idea of an equilibrium point. An **equilibrium point** of a state model is a point $(\vec{x}^*, \vec{u}^*)$ where the system remains at rest. This implies that $f(\vec{x}^*, \vec{u}^*) = \vec{0}$.

4.1 Pendulum Example

Let us revisit the pendulum system from the previous note and add a small torque T_{in} as an input.



For state variables $x_1 = \theta$ and $x_2 = \omega$, and input $u = T_{in}$ the state-space equations for the pendulum are[1]

$$\frac{d}{dt}\vec{x}(t) = f(\vec{x}(t)) = \begin{bmatrix} x_2(t) \\ -\frac{k}{m}x_2(t) - \frac{g}{\ell}\sin x_1(t) + T_{in}(t) \end{bmatrix}. \tag{9}$$

[1] ω is the angular velocity and is equivalent to $\frac{d\theta}{dt}$.

4.1.1 Equilbrium Analysis

For the system to be at equilibrium, x_2 must be equal to zero. Then we can break it down by cases depending on whether T is nonzero.

If we apply zero torque, (i.e. $u^* = 0$), then $\frac{g}{\ell}\sin x_1 = 0$ and there will be two equilibrium points for this system at: $(x_1, x_2) = (0,0)$ and $(\pi, 0)$. This is when the pendulum is at rest in a downward position or when it is turned upside down.

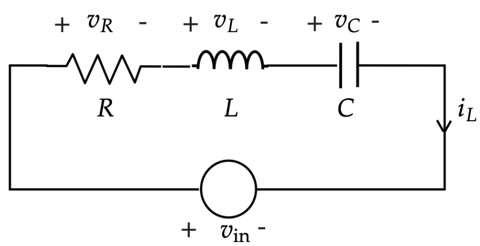
For a nonzero constant torque, $u^* = T > 0$ the second equation tells us that

$$\frac{g}{\ell}\sin x_1(t) = T$$

Therefore, there will be two equilibrium points at $(x_1, x_2) = (\sin^{-1}(\frac{\ell}{g}T), 0)$ and $(\pi - \sin^{-1}(\frac{\ell}{g}T), 0)$.

4.2 RLC Circuit

Consider the RLC circuit depicted on the right where $u = v_{in}$ denotes the input voltage.



Then picking state variables $x_1 = v_c$, $x_2 = i_L$ with input $u = v_{in}$, we can write the state-equations

$$\frac{d}{dt}\vec{x}(t) = f(\vec{x}(t), u(t)) = \begin{bmatrix} \frac{1}{C}x_2(t) \\ \frac{1}{L}\left(-x_1(t) - Rx_2(t) + u(t)\right) \end{bmatrix} \tag{10}$$

The equilibrium points for this system are when $x_1 = u$ and $x_2 = 0$. For a constant $u > 0$, this is when the RLC circuit has reached steady state and the capacitor is fully charged.

4.3 Taylor Expansion

What would happen if we expanded a non-linear function $f(\vec{x})$ around an equilibrium point (x^*)?

$$f_\ell(\vec{x}) = f(\vec{x}^*) + J_{\vec{x}}(\vec{x}^*) \cdot (\vec{x} - \vec{x}^*) = J_{\vec{x}}(\vec{x}) \cdot \vec{x}_\ell \tag{11}$$

At a first glance, we see that the constant term $f(x^*)$ goes to zero and we have a linear system $\frac{d}{dt}\vec{x} = A\vec{x}(t)$ where $A = J_{\vec{x}}$. From a physical perspective, the system remains at the equilibrium point $\vec{x}^*$ and $\vec{x}_\ell$ represents the amount of perturbation we make away from the equilibrium.

4.4 Equilibria for Linear Systems

As an aside, we analyze the equilibrium points of linear systems. This analysis should reinforce the idea that linear systems are "easy" to analyze and have predictable behavior in comparison to nonlinear systems.

Continuous-Time Systems

Let look at the conditions for a linear system represented by the differential equation

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t) \tag{12}$$

From the get-go we see that $(\vec{x}^*, \vec{u}^*) = (\vec{0}, \vec{0})$ must be an equilibrium point. This is since the system is at rest. Now if we put in a constant input $\vec{u}^*$ then to solve for equilibria, we get the following system of equations

$$A\vec{x} + B\vec{u}^* = \vec{0} \tag{13}$$

To solve for the states $\vec{x}$ in which the system would be in equilibrium, our analysis boils down to whether the square matrix A is invertible. [2]

1. If A is invertible, then there is a unique equilibrium point $\vec{x}^* = -A^{-1}B\vec{u}^*$.
2. If A is non-invertible, depending on the range of A, we have two scenarios.
   - If $B\vec{u} \in \text{Col}(A)$ then we will have infinitely many equilibrium points.
   - If $B\vec{u} \notin \text{Col}(A)$ then the system has no solution and we will have no equilibrium points.

Discrete-Time Systems

Now let's take a look at the discrete-time system

$$\vec{x}[t] = A\vec{x}[t] + B\vec{u}[t] \tag{14}$$

Again we see that $(\vec{0}, \vec{0})$ is an equilibrium point but notice that the conditions for equilibria are different for discrete-time systems. A system is in equilibrium if it is not changing. In otherwords, this means that $\vec{x}^*[t+1] = \vec{x}^*[t]$ therefore, for a constant input $\vec{u}^*$ we get the following system of equations

$$\vec{x} = A\vec{x} + B\vec{u}^* \implies (I - A)\vec{x} = B\vec{u}^* \tag{15}$$

The conditions for equilibria now depend on the matrix $I - A$ being invertible instead of the matrix A.

[2] This should be review from 16A/54, but we restate it here since it isn't quite obvious when A is singular or non-invertible. Normally a singular matrix has infinite solutions but take the system $A\vec{x} = \vec{b}$ with $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This leads to a contradiction that $x_1 = 0 \neq 1$.

5 Linearization

Now that we've looked at the Taylor Expansion of a function and the equilibrium points of a system, we develop the approach to linearizing a nonlinear system. Let's start by considering the 1D scalar case with state x and scalar input u.

$$\frac{d}{dt}x = f(x,u) \tag{16}$$

Expanding out f around the equilibrium point (x^*,u^*), we can create a linear system of the form

$$\begin{aligned} \frac{dx_\ell}{dt} &= f(x^*,u^*) + \frac{\partial f}{\partial x}(x^*,u^*)\cdot x_\ell + \frac{\partial f}{\partial u}(x^*,u^*)\cdot x_\ell & (17)\\ &= \alpha x_\ell + \beta u_\ell & (18) \end{aligned}$$

To extend this to the vector case let's consider a nonlinear system with state $\vec{x}$ and input $\vec{u}$

$$\frac{d}{dt}\vec{x} = f(\vec{x},\vec{u}) \tag{19}$$

To expand f around an equilibrium (x^*,u^*), we will need two Jacobians: one with partial derivatives with respect to $\vec{x}$ and another with partial derivatives with respect to $\vec{u}$

$$\frac{d}{dt}\vec{x}_\ell = A\vec{x}_\ell + B\vec{u}_\ell \tag{20}$$

Writing the Jacobians out explicitly, we see that

$$A = J_{\vec{x}}(\vec{x}^*,\vec{u}^*) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\Bigg|_{\vec{x}^*,\vec{u}^*} \tag{21}$$

$$B = J_{\vec{u}}(\vec{x}^*,\vec{u}^*) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial x_p} \end{bmatrix}\Bigg|_{\vec{x}^*\vec{u}^*} \tag{22}$$

Note that A and B will have different shapes. The matrix A is an $n \times n$ matrix while the B matrix will be an $n \times p$ matrix if the input $\vec{u}$ is length $p \times 1$ vector.

To summarize our work, we give the following steps to analyze a nonlinear system through Linearization.[3]

1 Write out the state-space model $\frac{d}{dt}\vec{x}(t) = f(\vec{x}(t),\vec{u}(t))$

2 Find an equilibrium point $\vec{x}^*,\vec{u}^*$ to analyze the system around.

3 Compute Jacobians of f, $J_{\vec{x}}$ and $J_{\vec{u}}$ and evaluate them at $\vec{x}^*,\vec{u}^*$.

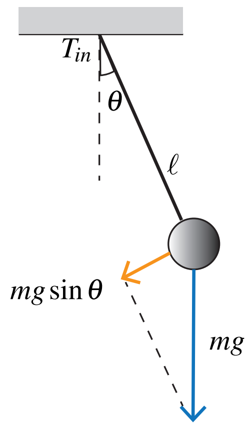
4 Defining variables $\vec{x}_\ell$ and $\vec{u}_\ell$ we can write out our linearized system as

$$\frac{d}{dt}\vec{x}_\ell = A\vec{x}_\ell + B\vec{u}_\ell \tag{23}$$

[3] We will only consider linearization for Continuous-Time Systems. The mathematics for linearizing Discrete-Time Systems is quite nuanced and is out of scope.

5.1 Linearized Pendulum with Torque

Now let's revisit the pendulum system that has a small torque T_{in} as an input to the system.



The new differential equation to this system is

$$m\ell\frac{d^2\theta(t)}{dt^2} + k\ell\frac{d\theta(t)}{dt} + mg\sin\theta(t) = \frac{T_{in}(t)}{\ell} \tag{24}$$

We can write out the state equations using the same state variables $x_1 = \theta$, $x_2 = \omega$ and input $u = T_{in}$

$$\frac{d}{dt}\vec{x} = f(\vec{x}(t), u) = \begin{bmatrix} x_2(t) \\ -\frac{k}{m}x_2(t) - \frac{g}{\ell}\sin x_1(t) + \frac{1}{\ell}u(t) \end{bmatrix} \tag{25}$$

Note that if we apply zero torque to the system, there are two equilibrium points at $\theta = 0$ and $\theta = \pi$.

The Jacobian matrices with respect to $\vec{x}$ and u are

$$J_{\vec{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \quad J_u = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} \tag{26}$$

Linearizing around the equilibrium point $x_1 = x_2 = u = 0$, the linearized system is

$$\frac{d}{dt}\vec{x}_\ell(t) = \begin{bmatrix} 0 & 1 \\ \frac{-g}{\ell} & \frac{-k}{m} \end{bmatrix}\vec{x}_\ell(t) + \begin{bmatrix} 0 \\ \frac{1}{\ell} \end{bmatrix} u_\ell(t) \tag{27}$$

Applying a small torque to the system, we can show that the system returns back to its equilibrium $\vec{x}^* = \vec{0}$.

Linearizing around the equilibrium point $x_1 = \pi$, $x_2 = 0$, $u = 0$, the linearized system is

$$\frac{d}{dt}\vec{x}_\ell(t) = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & \frac{-k}{m} \end{bmatrix}\vec{x}_\ell(t) + \begin{bmatrix} 0 \\ \frac{1}{\ell} \end{bmatrix} u_\ell(t) \tag{28}$$

Now if we now apply a small torque to the system, we see that the pendulum rapidly drops to its downard position. In a small neighborhood around $\theta = \pi$, we see that this inverted pendulum is *unstable*. This can also be explained by eigenvalues, but we will revisit the notion of stability later in the course.

6 Conclusion

As we've mentioned many times through the note, linear systems are desirable since they have closed form solutions and are easy to analyze. Systems in the real world however, will almost always be nonlinear due to external factors such as noise or distrubances. Therefore, we developed an approach called **Linearization** by leveraging Taylor Theorem and expanding out f around a small region about an **equilibrium point.**

From here onward, we will focus our study of control systems solely to linear models of the form $f(\vec{x}, \vec{u}) = A\vec{x} + B\vec{u}$ since a nonlinear system can be made linear by linearizing around its equilibria. In the next set of notes, we investigate into properties of linear systems that allow us to better understand and control them. A common theme throughout these notes will be analyzing the eigenvalues of the A matrix which is made possible since our system is linear.

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