EECS 16B Designing Information Devices and Systems II
Fall 2020 UC Berkeley **Note 14**

1 Overview

We have analyzed the stability of a system and its robustness to errors due to noise or distrubances. To do this, we used boundedness as a measure of how a system responds to external factors and claimed that a system was stable if the natural response was bounded for every bounded input. However, what if the system is unstable, does this mean all hope is lost? Fortunately there is a way to stabilize an unstable system using a technique called **feedback control.**

In this note, we explore how to stabilize unstable systems by placing the eigenvalues in the left-half of the complex plane. In addition, we will explore general system responses dictated by the eigenvalues to explain preferable design choices. Sometimes, we may want to use feedback-control on a stable system to remove any oscillations in the response.

Lastly, we will explore a special form in which it is easy to perform feedback control called **Controllable Canonical Form.** This will provide a further analysis on when we can perform feedback control and realize that this condition relies on the controllability of the system.

2 Feedback Control

Let us start by considering the following discrete-time system

$$\vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t] \tag{1}$$

To design a feedback controller, we can observe the state $\vec{x}[t]$ and then apply the input $\vec{u}[t] = K\vec{x}[t]$.

$$\vec{x}[t+1] = A\vec{x}[t] + BK\vec{x}[t] = (A+BK)\vec{x}[t] \tag{2}$$

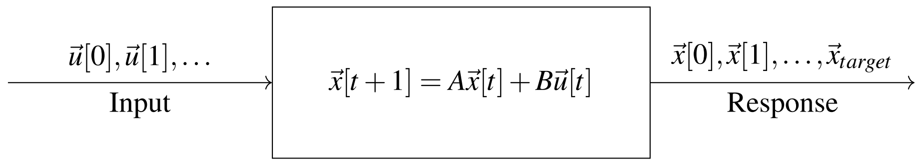
By applying this new input, the eigenvalues of the system will become the eigenvalues of the new matrix $A_{cl} = A + BK$. This technique is called **closed-loop control** since the effect of inputting $K\vec{x}[t]$ performs the action of negative feedback.[1]

Ideally, we would like to choose a K that moves the eigenvalues of the system into a desirable stable region. We will later show that for scalar input systems, we can pick values of k to assign the eigenvalues anywhere in the complex plane.

2.1 Comparison to Open Loop Control

Recall that controllability allowed us to calculate an input sequence $\vec{u}[0], \vec{u}[1], \vec{u}[2], \ldots$ that drives the state from $\vec{x}[0]$ to any $\vec{x}_{target}$. Thus, an alternative to the feedback control is to select $\vec{x}_{target} = \vec{0}$, calculate an input sequence based on $\vec{x}[0]$, and to apply this sequence in an "open-loop" fashion without using further state measurements as depicted below.

[1] Standard controls literature uses $\vec{u} = -K\vec{x}$ and the closed-loop matrix is referred to as $A_{cl} = A - BK$.



The trouble with this open-loop approach is that it is sensitive to uncertainties in A and B, and does not make provisions against disturbances that may act on the system. By contrast, feedback offers a degree of robustness: if our design of K brings the eigenvalues of $A+BK$ to well within the unit circle, then small perturbations in A and B would not move these eigenvalues outside the circle. Thus, despite the uncertainty, solutions converge to $\vec{x} = \vec{0}$ in the absence of disturbances and remain bounded in the presence of bounded disturbances.

2.2 Examples

2.2.1 Scalar System

Let's take a look at a scalar system with some error $w[t]$ represented as

$$x[t+1] = ax[t] + bu[t] + w[t] \tag{3}$$

Applying a feedback input $u[t] = kx[t]$, the closed-loop scalar system will be

$$x[t+1] = (a+bk)x[t] + w[t] \tag{4}$$

If our original system was unstable meaning $|a| \geq 1$, we could pick a value of k that makes $|a+bk| < 1$ stabilizing the system. Note that if we picked $k = -\frac{a}{b}$, then the eigenvalue of the system becomes 0 but we are still unable to remove the error term $w[t]$.

2.2.2 2D System

Now let's observe the a system in the vector case represented by the differential equation

$$\vec{x}[t+1] = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[t] \tag{5}$$

This system has eigenvalues 3 and -1 meaning it is unstable since both eigenvalues have magnitude greater than or equal to 1. Let's try to use feeback control to stabilize this system by letting our unknown K be

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \tag{6}$$

With the feedback law $u[t] = K\vec{x}[t]$, let's see where we can place the eigenvalues of $A+BK$

$$A+BK = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} \tag{7}$$

The characteristic polyonmial and eigenvalues can be computed accordingly

$$(\lambda)(\lambda - 2 - k_2) - (-3 + k_1) = 0 \implies \lambda^2 - (k_2+2)\lambda - (k_1 - 3) = 0. \tag{8}$$

Notice how we have control over both coefficients of the characteristic polynomial! This implies that we can always choose our eigenvalues by picking the correct values of k_1 and k_2. [2]

For example, if we would like the eigenvalues to be at $\lambda = -\frac{1}{2}, \frac{1}{2}$, then we require the characteristic polynomial

$$(\lambda - \frac{1}{2})(\lambda + \frac{1}{2}) = \lambda^2 - \frac{1}{4} \implies k_1 = \frac{13}{4}, k_2 = -2 \tag{9}$$

Note that in order to assign the eigenvalues arbitrary, the system **must be controllable.** We will show why this is the case in a later section by showing that we can always change coordinates into a space in which we can assign eigenvalues arbitrarily if and only if the system is controllabe.

2.2.3 Uncontrollable System

So what would happen if our system was uncontrollable meaning we cannot to reach everywhere in our state-space? The next example shows that we can still perform feedback control, but our choice of eigenvalues will be limited due to uncontrollability.

Let us consider the uncontrollable and unstable system

$$\vec{x}[t+1] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[t] \tag{10}$$

We can verify that the system is uncontrollable by computing the controllability matrix

$$\mathscr{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \implies \text{Rank}\,\mathscr{C} = 1 < 2 \tag{11}$$

However, this does not stop us from inputting a feedback law into our system $u[t] = K\vec{x}[t]$. Therefore, let us apply this feedback and look at the eigenvalues of the closed-loop system.

$$A + BK = \begin{bmatrix} 1 & 0 \\ 1+k_1 & 1+k_2 \end{bmatrix} \implies \lambda^2 - \lambda + (1+k_2) = 0 \tag{12}$$

Note how the k_1 term has disappeared from the characteristic polynomial. This should come from the intuition that we are not fully able to control our system and some of our control input is lost due to the nature of the system.

We still have one degree of freedom in picking k_2 to assign our eigenvalues. However, notice how we cannot control the coefficient of the λ term meaning our eigenvalues will always be of the form

$$\lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4(1+k_2)} \tag{13}$$

Note that if this were a continouous-time system, we would be unable to stabilize the system. However, since this is a discrete-time system, we can still stabilize it with a repeated eigenvalue $\lambda = \frac{1}{2}$ by picking $k_2 = -\frac{3}{4}$.

[2] Note that in practice, we may have constraints on how much we can amplify our state vector $\vec{x}$.

3 System Responses

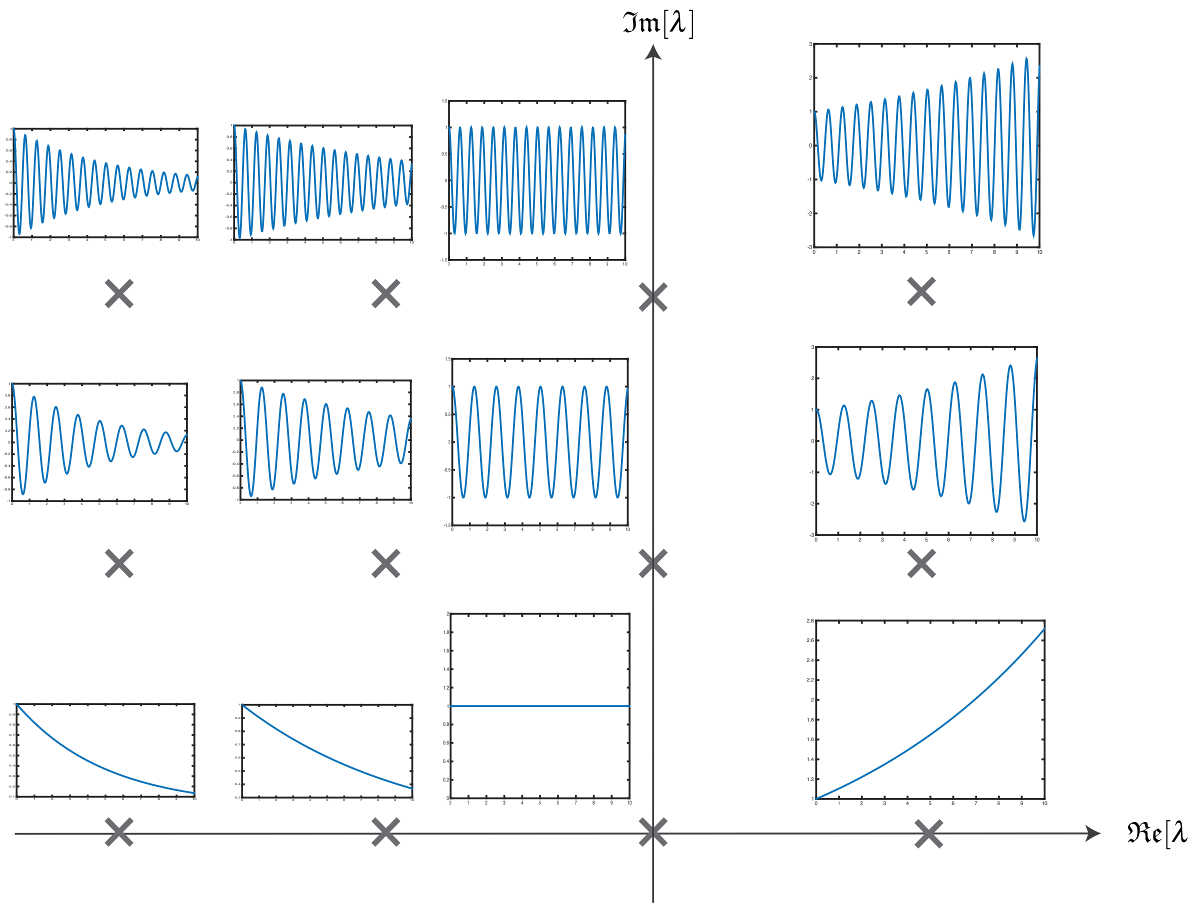
Now that we understand how to place our eigenvalues using feedback control, let's analyze the natural response of a system based on its eigenvalues to help us make our design choices.

3.1 Continuous-Time Responses

Recall that a continuous-time system is stable if all of its eigenvalues have negative real part. This means our eigenvalues should be in the left-half of the complex plane to guarantee stability.

However, just being in the left-half plane does not always mean our response is desirable. If the imaginary component of the eigenvalues are really large, then we will see huge oscillations.

The visual below depicts the real part of $e^{\lambda t}$ for a given λ



This should show that $e^{\lambda t}$ is oscillatory when λ has an imaginary component. The exponential grows unbounded when $\mathfrak{Re}[\lambda] > 0$, decays to zero for $\mathfrak{Re}[\lambda] < 0$ and has constant amplitude when $\mathfrak{Re}[\lambda] = 0$.

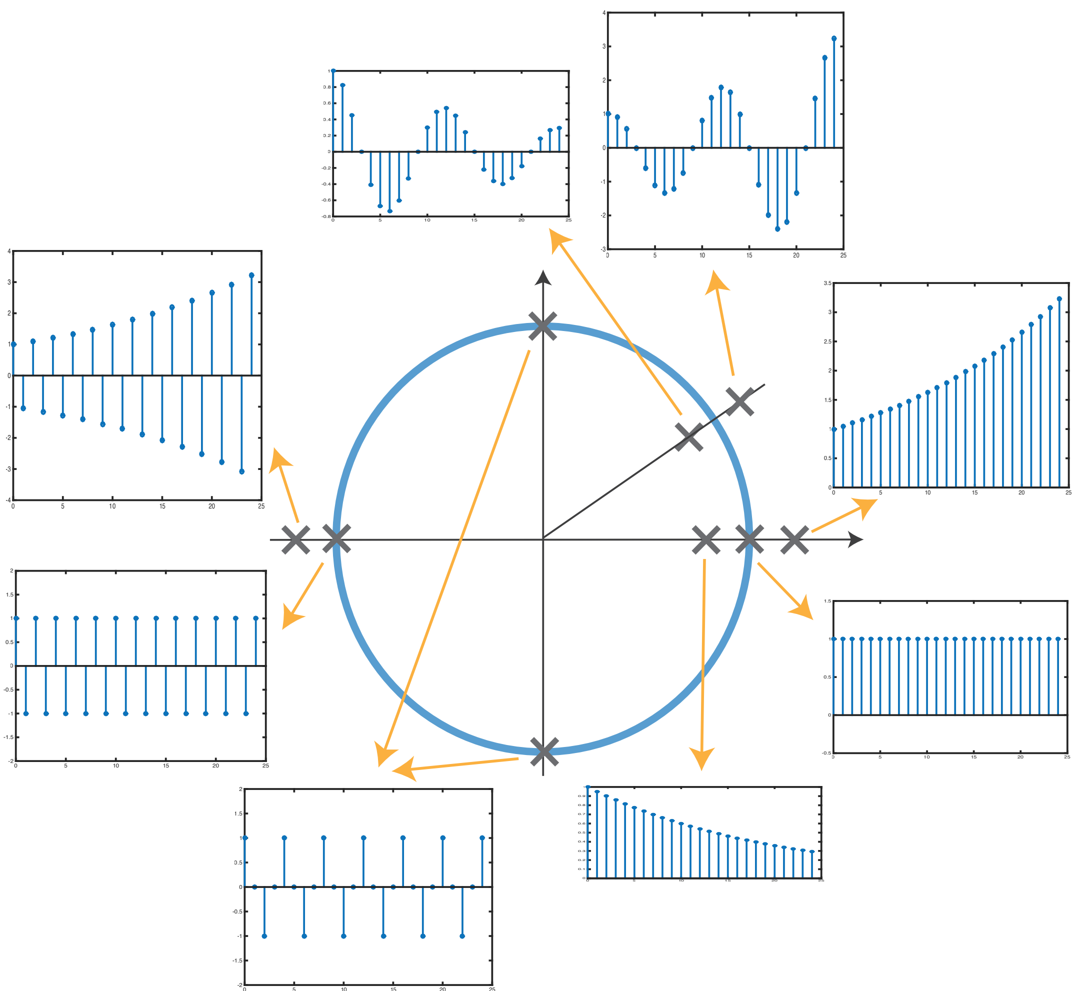
From a controls perspective, we want to reach our target input $u(t)$ as quickly as we can with minimal oscillations. Having $\mathfrak{Re}[\lambda] < 0$ ensures stability and $\mathfrak{Im}[\lambda] = 0$ prevents oscillations.

However, removing oscillation comes at the cost of taking longer to reach our target. As a result, if we would like to reach our target at a quicker rate, there will be a tradeoff between oscillations and the rise time of the system. If you take EE 128, you will learn how to design these types of controllers.

3.2 Discrete-Time Responses

Now let's analyze the discrete-time case. Recall that all of the eigenvalues must be inside the unit circle in order to guarantee stability.

The visual below depicts the real part of λ^t for a given λ



Oscillations in the discrete-time case are a bit trickier to understand, but we can see that if λ is purely real and positive, then our response will be oscillation free. The response grows unbounded when $|\lambda| > 1$, decays to zero for $|\lambda| < 1$ and has constant amplitude for $|\lambda| = 1$.

In order to reach our target input $u[t]$ quickly without oscillations, we should pick a real, positive, λ less than 1. Notice how if λ is negative, the response is oscillatory since $(-1)^t$ alternates between positive and negative values. [3]

[3] If you take EE120, you will learn that $(-1)^t$ has a frequency of π since $(-1)^t = e^{j\pi t}$. To avoid any oscillations, we should pick eigenvalues with zero frequency similar to what we did for the Continuous-Time case.

4 Controllable Canonical Form (Out of Scope)

In this section, we will prove that if a system is controllable, we are able to assign its eigenvalues arbitrarily.

The key to this argument is the development of a special system with the matrix A and vector B represented as

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} \end{bmatrix} \qquad B_c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{14}$$

This representation is called **Controllable Canonical Form** since the resulting model is guaranteed to be controllable.

4.1 Eigenvalues in CCF

Let us compute the eigenvalues of a system that is in Controllable Canonical Form.

$$\det(\lambda I - A) = \lambda^n - a_{n-1}\lambda^{n-1} - a_{n-2}\lambda^{n-2} - \cdots - a_1\lambda - a_0 \tag{15}$$

Note how the coefficients of the characteristic polynomial align perfectly with the bottom row of the A_c matrix.

Now what if we wanted to peform feedback control? Let us look at the eigenvalues of $A_c + B_c K$ where

$$K = \begin{bmatrix} k_0 & k_1 & \cdots & k_{n-2} & k_{n-1} \end{bmatrix} \tag{16}$$

The closed-loop matrix in canonical form A_{cl} would be

$$A_{cl} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_0 + k_0 & a_1 + k_1 & a_2 + k_2 & \dots & a_{n-1} + k_{n-1} \end{bmatrix} \tag{17}$$

Therefore, the characteristic polynomial of the closed-loop matrix will be

$$\det(\lambda I - A_{cl}) = \lambda^n - (a_{n-1} + k_{n-1})\lambda^{n-1} - (a_{n-2} + k_{n-2})\lambda^{n-2} - \dots - (a_1 + k_1)\lambda - (a_0 + k_0) \tag{18}$$

Note how each k_i appears in exactly one coefficient and we can change it to its desired value. This is why eigenvalue placement is very easy in Controllable Canonical Form.

4.2 Controller Basis

So how do we prove that for any controllable system

$$\vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t] \tag{19}$$

that we can assign the eigenvalues of $A+BK$ arbitrarily? We will do this by showing changing coordinates into a basis in which the system is in Controllable Canonical Form.

Note that we are not proving its existence, rather we are first showing what the transformation to the controller basis is, given that it exists. To start, let's assume there exists some change of variables $\vec{z} = T\vec{x}$ that brings A and B into CCF. This would imply that

$$A_c = TAT^{-1} \qquad B_c = TB \tag{20}$$

If this were the case, we would be able to design a feedback controller $u = K_c\vec{z}$ to assign the eigenvalues of $A_{cl} = A_c + B_cK_c$ arbitrarily. Since $\vec{z} = T\vec{x}$, $u = K_cT\vec{x}$ it follows that

$$K = K_cT \implies K_c = KT^{-1} \tag{21}$$

Note that $A_{cl} = A_c + B_cK_c = T(A+BK)T^{-1}$ meaning the eigenvalues of A_{cl} are identical to the eigenvalues of $A+BK$ which was our original goal. This results from the fact that changing coordinates of a system does not change its eigenvalues. [4]

4.3 Existence of CCF

We will now show what the matrix T that brings a system into Controllable Canonical Form is. Let's start by assuming the original system (19) is controllable. Then the controllability matrix

$$C = \begin{bmatrix} \vec{b} & A\vec{b} & \cdots & A^{n-1}\vec{b} \end{bmatrix} \tag{22}$$

must be full-rank, square, and invertible. Therefore, let $\vec{q}^T$ denote the first row of the inverse C^{-1}. It follows from the identity $C^{-1}C = I$ that

$$\vec{q}^T C = \begin{bmatrix} \vec{q}^T\vec{b} & \vec{q}^T A\vec{b} & \cdots & \vec{q}^T A^{n-1}\vec{b} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix} \tag{23}$$

We will use this equation to show that the choice

$$T = \begin{bmatrix} \vec{q}^T \\ \vec{q}^T A \\ \vdots \\ \vec{q}^T A^{n-1} \end{bmatrix} \tag{24}$$

is indeed the transformation that gets us into Controllable Canonical Form.

[4] We can show this mathematically since $\det(TAT^{-1} - \lambda I) = \det(TAT^{-1} - T(\lambda I)T^{-1}) = \det(T(A-\lambda I)T^{-1}) = \det(A - \lambda I)$.

We can first verify that $B_c = T\vec{b}$ since

$$T\vec{b} = \begin{bmatrix} \vec{q}^T \\ \vec{q}^T A \\ \vdots \\ \vec{q}^T A^{n-1} \end{bmatrix} \vec{b} = \begin{bmatrix} \vec{q}^T \vec{b} \\ \vec{q}^T A\vec{b} \\ \vdots \\ \vec{q}^T A^{n-1}\vec{b} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{25}$$

To show that $A_c = TAT^{-1}$, note that

$$TA = \begin{bmatrix} \vec{q}^T \\ \vec{q}^T A \\ \vdots \\ \vec{q}^T A^{n-1} \end{bmatrix} A = \begin{bmatrix} \vec{q}^T A \\ \vec{q}^T A^2 \\ \vdots \\ \vec{q}^T A^n \end{bmatrix} \tag{26}$$

and compare this to

$$A_c T = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & \ldots & a_{n-1} \end{bmatrix} \begin{bmatrix} \vec{q}^T \\ \vec{q}^T A \\ \vdots \\ \vec{q}^T A^{n-1} \end{bmatrix} = \begin{bmatrix} q^T A \\ \vec{q}^T A^2 \\ \vdots \\ \vec{q}^T \left(a_0 I + a_1 A + \ldots + a_{n-1} A^{n-1}\right) \end{bmatrix} = \begin{bmatrix} q^T A \\ \vec{q}^T A^2 \\ \vdots \\ \vec{q}^T A^n \end{bmatrix} \tag{27}$$

The equality from the last row follows from the Cayley-Hamilton Theorem which states that

$$A^n - a_{n-1}A^{n-1} - \ldots - a_1 A - a_0 I = 0 \tag{28}$$

We can also show that the matrix T can be written as the following

$$TC = \begin{bmatrix} T\vec{b} & TA\vec{b} & \ldots & TA^{n-1}\vec{b} \end{bmatrix} = \begin{bmatrix} B_c & A_c B_c & \ldots & A_c^{n-1} B_c \end{bmatrix} = \widetilde{C} \tag{29}$$

$$\implies T = \widetilde{C} C^{-1} \tag{30}$$

where $\widetilde{C}$ represents the controllability matrix in the controller basis.

4.4 Differential Equations

As an aside, this section will explore the connections between n^{th} order differential equations and Controllable Canonical Form. Let's start with the n^{th} order differential equation with a scalar input

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_1\frac{dy}{dt} + a_0 y = u(t) \tag{31}$$

By picking state variables $x_1 = y, x_2 = \frac{dy}{dt}, x_{n-1} = \frac{d^{n-2}y}{dt^{n-2}}, x_n = \frac{d^{n-1}y}{dt^{n-1}}$, we can create the following vector differential equation

$$\frac{d}{dt}\vec{x}(t) \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t) \tag{32}$$

Notice how this system is in Controllable Canonical Form! Its characteristic polynomial will be

$$\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 \tag{33}$$

which is almost identical to our original differential equation. This is because $e^{\lambda t}$ is an eigenfunction of the differentiation operator and plugging in $y(t) = e^{\lambda t}$ into the differential equation will in fact give the same characteristic polynomial.

When we were looking for the eigenvalues of the system, we were in fact looking for the eigenvalues of the differentiation operator. Naturally the two approaches of solving an n^{th} order differential equation and solving the vector differential equation are one of the same.

This connection to differential equations should give more meaning to the canonical nature of Controllable Canonical Form. Looking at the individual states, we see a chain of integrators where

$$\frac{d}{dt}x_1(t) = x_2(t) \qquad \frac{d}{dt}x_2(t) = x_3(t) \qquad \ldots \qquad \frac{d}{dt}x_{n-1}(t) = x_n(t) \tag{34}$$

$$\implies x_1(t) = \int x_2(t) \qquad x_2(t) = \int x_3(t) \qquad \ldots \qquad x_{n-1}(t) = \int x_n(t) \tag{35}$$

There is another underlying connection between state-space representation and the transfer function of a system which gives more meaning to this idea of chaining integrators. You can learn more about this idea in courses like EE 120 and 128.

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