EECS 16B Fall 2020 | Designing Information Devices and Systems II UC Berkeley | Note 2

1 Introduction

In the previous note, we looked into how fast a computer can operate and how much energy it would take. This was done by modeling transistors as switch with a resistor and gate capacitance and through the mathematics of differential equations.

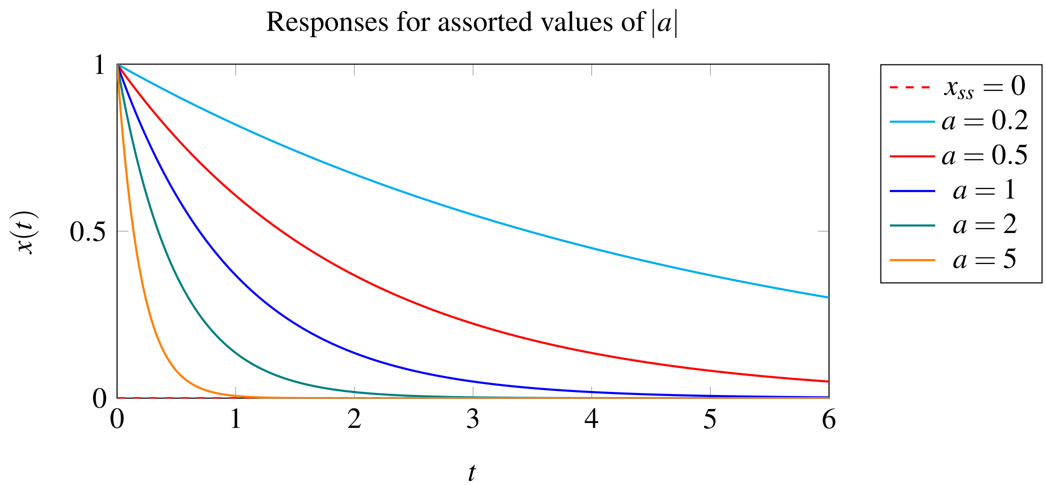
Now that we understand differential equations, we will try to understand what exactly controls the speed of our computers. In addition, we will look at piecewise constant inputs which will give us some insight on how we can control our systems.

2 Time Constants

The differential equation $\frac{d}{dt}x(t) = \lambda x(t)$ with initial condition $x(0) = x_0$ has solution $x(t) = x_0 e^{\lambda t}$. Note that when we analyzed the CMOS inverter, the constant λ was always negative.

This implies that the solution to an RC circuit will always be a decaying exponential since the physical values of the resistor and capacitor are always positive. Notice that as $t \to \infty$, the response decays to 0 and the value of $a = |\lambda|$ dictates how long it takes our response to reach steady state.

We provide some plots below to illustrate this effect with multiple values of a.



Notice how the response decays quicker for larger values of a. This quantity is so important, that we call $\frac{1}{a}$ the **time constant** denoted by the Greek letter τ. Mathematically, it is defined as the time it takes for the response to be within $\frac{1}{e}$ of its steady state value.

For an exponential that decays to 0, this would be the time at which it decays to $\frac{1}{e} = 36.8\%$ of its initial value whereas for a rising exponential, this would the time at which the response rises to $1 - \frac{1}{e} = 63.2\%$ of its steady state value.

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Looking back at our RC circuit modeled by the differential equation $\frac{d}{dt}v(t) = -\frac{1}{RC}v(t)$, with initial condition $v(0) = V_{DD}$, we can solve for the time constant by finding the time τ at which $v(\tau) = \frac{V_{DD}}{e}$.

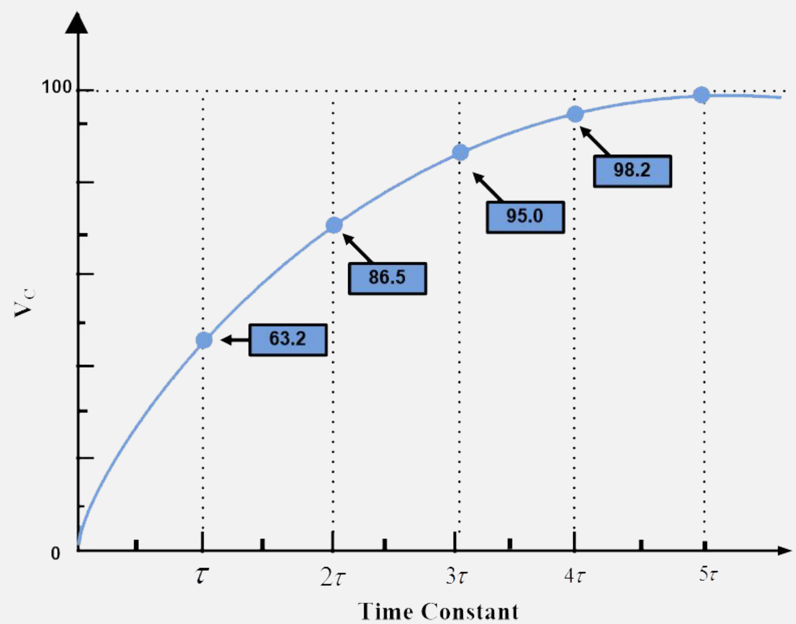
$$v(\tau) = V_{DD}e^{-\frac{\tau}{RC}} = \frac{V_{DD}}{e}$$

A quick computation tells us that our time constant $\tau = RC$. This immediately tells us that the physical values of the resistor and capacitor are what affect the speed at which our response decays to 0 or rises to V_{DD}. If we wanted to speed up the response by lowering τ, we would have to either lower the value of our resistor or capacitor.

We've shown a diagram below as a reference depicting how much a capacitor charges after a certain number of time constants. From the diagram we see that after 3τ, the capacitor has charged up to 95% and after 5τ the response will be within 1% of its steady state value.

How many τ will it take?

With our new definition of a time constant τ we have not only understood how long it takes for our differential equation to reach steady state, but we have also created a metric by which we can measure how close to steady state our response will be after a specified period of time.



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3 Piecewise Constant Time Varying Inputs

Let's consider the following RC circuit with a "input" source $v_s(t)$ that changes over time.

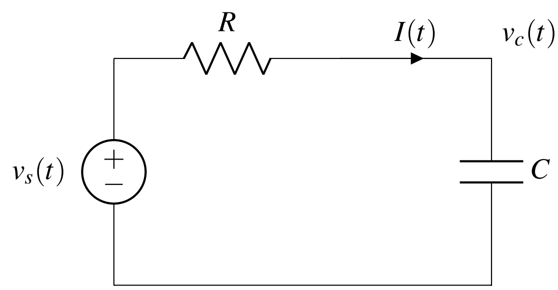


Figure 1: The source $v_s(t)$ changes over time.

In the previous note, we learned to solve for the transient voltage $v_c(t)$ on a capacitor when $v_s(t) = 0$ and $v_s(t) = V_{DD}$. For example, when $v_s = V_{DD}$, we wrote the differential equation

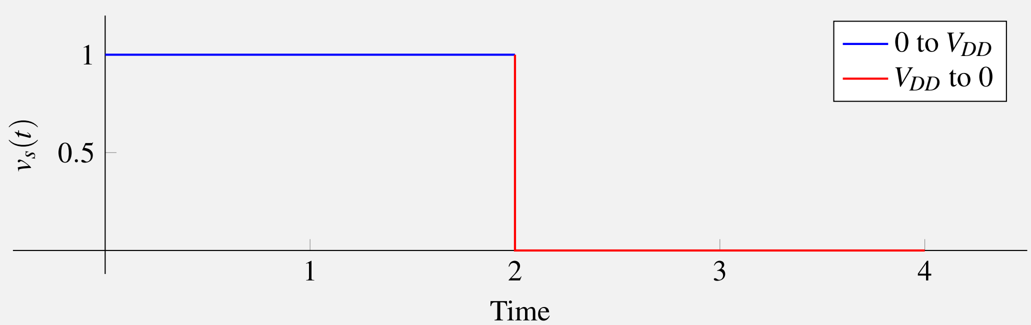
$$\frac{d}{dt}v_c(t) = -\frac{v_c(t)}{RC} + \frac{V_{DD}}{RC}$$

and arrived at the following solution for $t \geq 0$.

$$v(t) = V_{DD}(1 - e^{-\frac{t}{RC}}).$$

In this note, we'll consider the case when $v_s(t)$ is piecewise constant. To solve these differential equations, we look at the differential equation in "windows" and find the solution for each window. We provide an illustration of what it means to be piecewise constant in the figure below.

Illustration of Piecewise Constant Inputs



We provide a piecewise constant example of $v_s(t)$ where $v_s(t) = 1$ for $t \in [0,2)$ and $v_s(t) = 0$ for $t \in [2,4)$. To solve for $v_c(t)$ we can treat $v_s(t)$ as a constant 1 or 0 depending on the interval we are in.

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4 Switching Inputs

Let us start by considering the most basic changing input that we can think of: A voltage turning on to some value V_{DD} and then turning off. We will assume that v_s has been 0 for a long time meaning $v_c(0) = 0\,\mathrm{V}$.

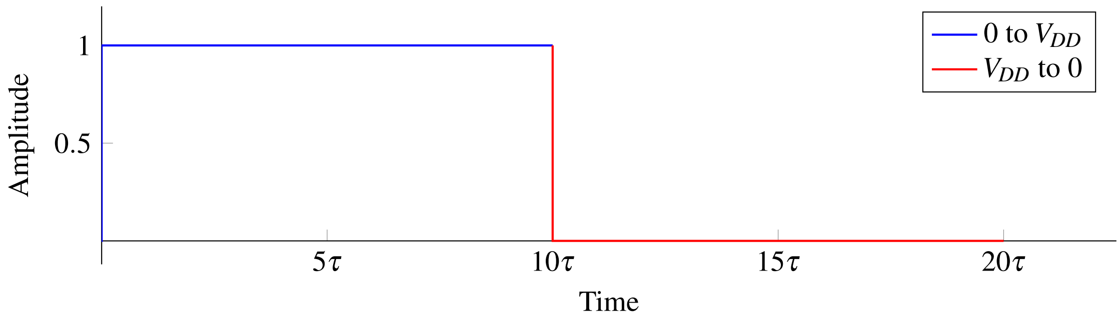


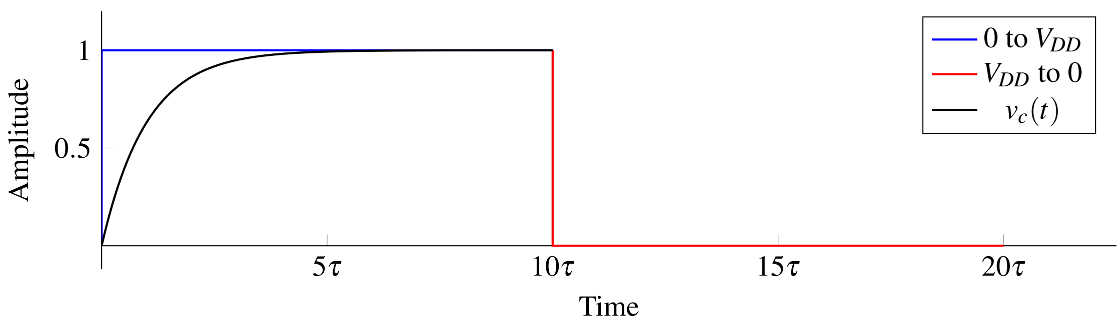
Figure 2: On and Off input: On for 10τ. Here $\tau = RC$ is the time constant for the circuit.

To solve for $v_c(t)$, we will break our problem into two by looking at the window when $v_s = V_{DD}$ and $v_s = 0$.

- $t \in [0, 10\tau)$: In this window, v_s remains constant at V_{DD}. Therefore, the differential equation will be

$$\frac{d}{dt}v_c(t) = -\frac{1}{RC}v_c(t) + \frac{V_{DD}}{RC}$$

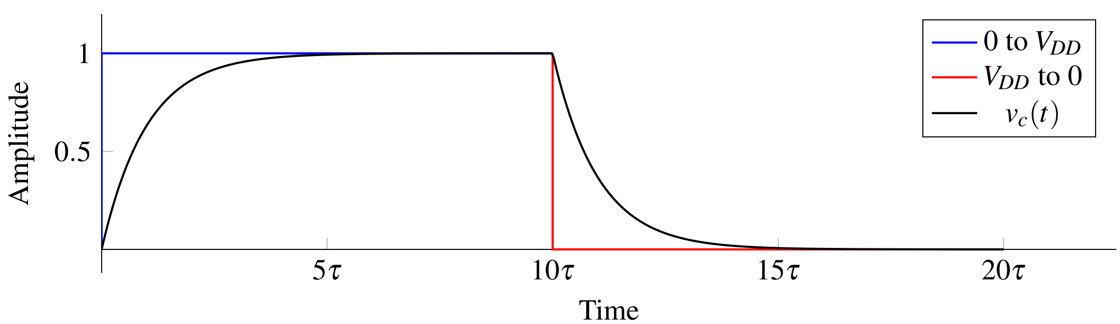
which has solution $v_c(t) = V_{DD}(1 - e^{-t/RC})$. Note that this solution is only valid for $t \in [0, 10\tau)$.



- $t \in [10\tau, 20\tau)$: In this window, v_s remains constant at $0\,\mathrm{V}$. Therefore, the differential equation will be

$$\frac{d}{dt}v_c(t) = -\frac{1}{RC}v_c(t)$$

The initial condition will be $v_c(10\tau) \approx 1\,\mathrm{V}$. After 10τ we can approximate v_c as fully charged. This tells us that the solution to the differential equation is $v_c(t) = V_{DD}e^{(t-10\tau)/RC}$.



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5 Another Example

Let us look at the exact same example, where v_s switches much quicker than the first example.

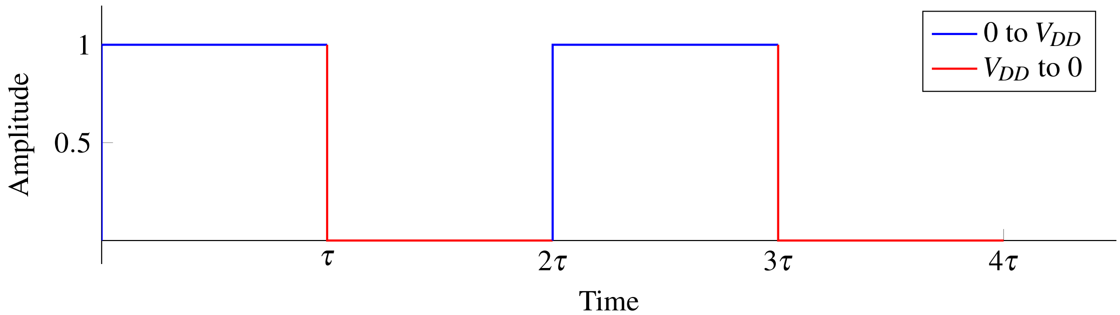
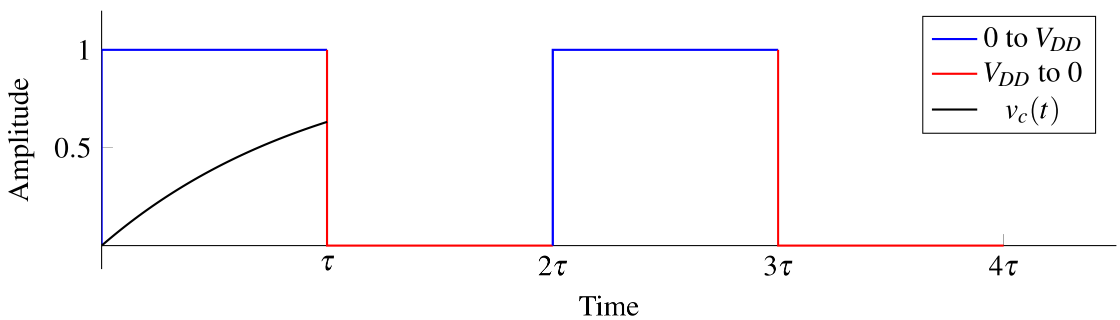


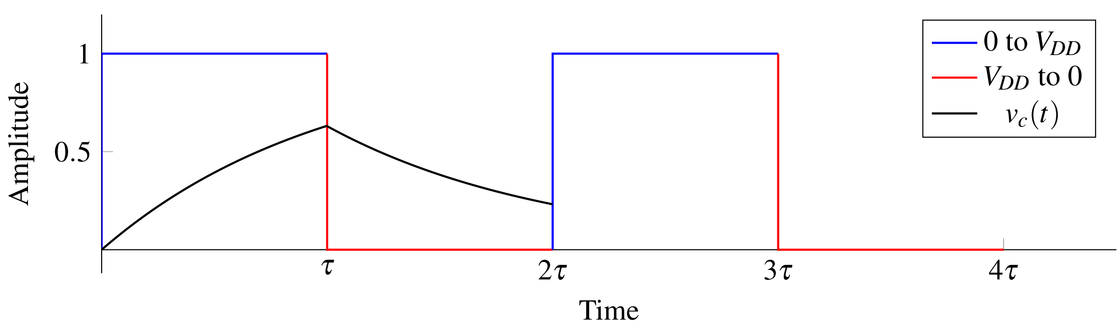
Figure 3: On and Off input: On for τ. Here $\tau = RC$ is the time constant for the circuit.

We can take a similar approach by look at each individual window where v_s is constant.

- $t \in [0, \tau)$: In this window, v_s remains constant at V_{DD}. Recall from the previous example that the solution is of the form $v_c(t) = V_{DD}(1 - e^{-t/RC})$. Notice how v_c only reaches 63% of V_{DD}.



- $t \in [\tau, 2\tau)$: In this window, v_s remains constant at 0 V. However, the initial condition will be $v_c(\tau) = 0.63\,\text{V}$. We can solve this differential equation to get solution $v_c(t) = 0.63e^{(t-1)/RC}$.



Again notice that v_c is unable to decay to zero since v_s switches every τ.

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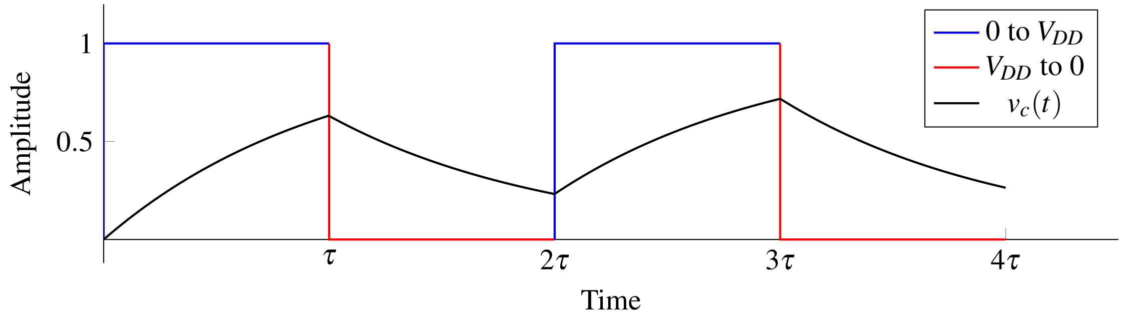
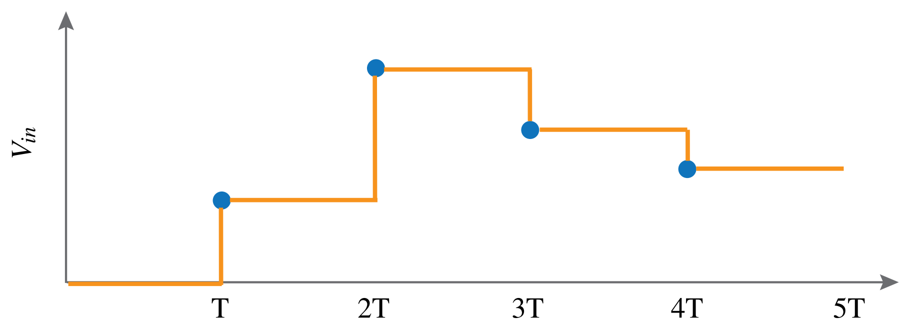
We can repeat the same process for the next two windows and we'll see that v_c is stuck in between 0 and 1.



This should give a better illustration of the effect that the time constant has on a first-order system.

Here's another interesting example to look at



You should notice that the solution $v_c(t)$ continually tries to follow the value of V_{in}. This is the key idea behind digital control and we will explore more of this in a later note.

Try creating your own piecewise inputs and analyze the effects of smaller and larger τ.

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