EECS 16B Designing Information Devices and Systems II
Fall 2020 UC Berkeley

Note 4

1 Introduction

In the previous sets of notes, we've developed a set of tools to solve a **scalar** differential equation of the form

$$\frac{d}{dt}x(t) = \lambda x(t) + u(t) \tag{1}$$

While scalar differential equations are very useful in modeling a first-order system such as an RC circuit, we will need to develop a new set of methodologies to tackle higher order differential equations.

Such differential equations are called **vector** differential equations and can be written in the form

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b} \tag{2}$$

In this note, we will focus on how to set-up and solve vector differential equations.

2 Notation

The vector $\vec{x}$ is often called the **state vector** and the individual entries $x_1, \ldots, x_n$ are called the **states.**

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

The matrix A is called the **differentiation matrix** since it performs the act of differenation.

3 Second Order System

Let's take a look at a more complicated RC circuit example with two resistors and capacitors.

Let $C_1 = C_2 = 1\mu F$, $R_1 = \frac{1}{3}\text{M}\Omega$ and $R_2 = \frac{1}{2}\text{M}\Omega$.

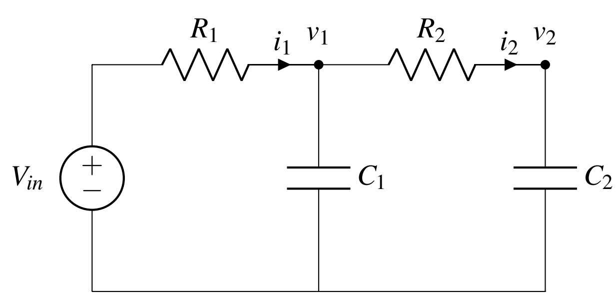


Figure 1: Two dimensional system

Now let's try to analyze the discharging case. Let $V_{in} = 1V$ for time $t < 0$, and $V_{in} = 0V$ for time $t \geq 0$. With this in mind, we have two steady state conditions:

(a) Initial condition $t = 0$: The voltage has been charging the capacitors for an infinite amount of time. Hence, both capacitors have voltage $v_{C_1} = v_{C_2} = 1V$ and the current $I_1 = I_2 = 0A$.

(b) As $t \to \infty$: After the capacitors have been allowed to discharge for a long period of time, they carry no charges on their plates, hence $v_{C_1} = v_{C_2} = 0V$.

Next, let us solve for the transients, i.e. how does our system go from (a) to (b)? First we need to set up the circuit equations.

$$v_2 = v_1 - i_2 \cdot R_2 \quad i_2 = C_2 \frac{d}{dt} v_2 \tag{3}$$

$$0 - v_1 = i_1 \cdot R_1 \Rightarrow i_1 = -\frac{v_1}{R_1} \tag{4}$$

$$i_1 = i_2 + C_1 \frac{d}{dt} v_1 \tag{5}$$

Next, to solve for the transients, we need to first define our system variables. **The standard approach that we will always take is to make anything that gets differentiated into a state variable.** Hence, we will need two state variables, v_1 and v_2, the voltages across C_1 and C_2 respectively. We will need to setup differential equations to solve for our system variables.

3.1 Systems of Differential Equations

Using the circuit equations above, we are able to isolate each derivative term and write them in terms of the state-variables.

$$\frac{d}{dt} v_1(t) = -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right) v_1(t) + \frac{v_2(t)}{R_2C_1} \tag{6}$$

$$\frac{d}{dt} v_2(t) = \frac{v_1(t)}{R_2C_2} - \frac{v_2(t)}{R_2C_2} \tag{7}$$

Now let us define our state-vector $\vec{x} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. This will yield vector-differential equation

$$\frac{d}{dt}\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right) v_1 + \frac{v_2}{R_2C_1} \\ \frac{v_1}{R_2C_2} - \frac{v_2}{R_2C_2} \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right) & \frac{1}{R_2C_1} \\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \tag{8}$$

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \vec{x}(t) \tag{9}$$

In equation (9), we have substituted in for the component values defined above so that we get a matrix with concrete numbers in it.

4 The Plan

Now that we have a vector-differential equation, we will outline the overall roadmap of this note.

1 First we will look at the case when A is diagonal and understand why it is easier to solve.

2 Then we will introduce a Linear Algebra technique called **diagonalization** to write A in terms of its eigenvectors and eigenvalues.

3 Lastly, we will solve the differential equation using the results from diagonalization.

4.1 Diagonal System

Let us consider the following circuit:

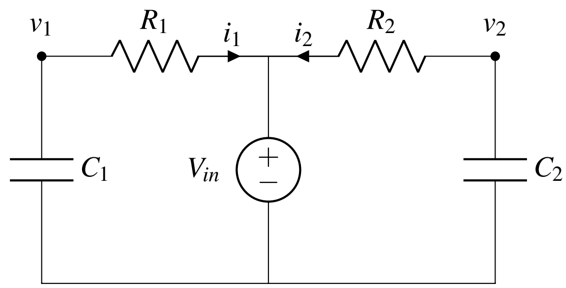


Figure 2: Diagonal System

Both the capacitors have been charged to v_{in} and at $t = 0$, we set $v_{in} = 0V$, and allow the capacitors to discharge. Hence our initial conditions are $v_1(0) = v_2(0) = V_{in}$. We get the following branch equations:

$$i_1 = -C_1 \frac{d}{dt} v_1 = \frac{v_1}{R_1} \tag{10}$$

$$i_2 = -C_2 \frac{d}{dt} v_2 = \frac{v_2}{R_2} \tag{11}$$

Hence from equations (10) and (11), we get the following uncoupled differential equation:

$$\frac{d}{dt} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1 C_1} & 0 \\ 0 & -\frac{1}{R_2 C_2} \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \tag{12}$$

Notice how the states v_1 and v_2 are independent of each other! This means we can solve for v_1 and v_2 using all of the techniques from the previous notes on scalar differential equations.

4.2 Diagonalization

Let's suppose we had a system of differential equations of the form

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) \tag{13}$$

and the matrix A is 2×2 with a set of linearly independent eigenpairs $(\lambda_1, \vec{v}_1)$ and $(\lambda_2, \vec{v}_2)$. [1]

Then by the definition of eigenvectors and eigenvalues, we know that

$$A\vec{v}_1 = \lambda_1 \vec{v}_1 \tag{14}$$

$$A\vec{v}_2 = \lambda_2 \vec{v}_2 \tag{15}$$

These two relationships can be expressed simultaneously using matrices that consolidate the eigenvectors (side by side) and eigenvalues (on a diagonal):

$$A\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{16}$$

Calling the former two matrices V and the latter Λ,

$$AV = V\Lambda \tag{17}$$

Because we chose two linearly independent eigenvectors to constitute V, V is invertible. Stating A in terms of its eigenvectors and eigenvalues is called the **eigendecomposition** or **diagonalization** of A :

$$A = V\Lambda V^{-1} \tag{18}$$

5 How to Solve?

Now that we've understood the motivation behind diagonal systems and on the **diagonalization** of a matrix A, we can go back to our original system of differential equations introduced in the first section.

Coming back to our original system, from equation (9),

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\vec{x}(t) \tag{19}$$

As discussed, let's use our diagonalization technique to solve this system of differential equations.

We first compute the eigenvectors and eigenvalues of the matrix A.

$$\lambda_1 = -6, \quad \vec{v}_1 = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \qquad \lambda_2 = -1, \quad \vec{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

[1] Note that this is not always the case and there are matrices that do not have a full set of eigenvectors that form a basis for $\mathbb{R}^n$. Such matrices are called defective matrices and we will explore them in the next note.

Define the matrix $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$, and the diagonal matrix $\Lambda = \begin{bmatrix} -6 & 0 \\ 0 & -1 \end{bmatrix}$, we can write $A = V\Lambda V^{-1}$ to rewrite equation (9) as follows:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) = V\Lambda V^{-1}\vec{x}(t) \tag{20}$$

$$= \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -6 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}^{-1} \vec{x}(t) \tag{21}$$

We will now define a new variable $\vec{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = V^{-1}\vec{x}(t)$. By left multiplying V^{-1} on both sides of equation (20), we get the following:

$$V^{-1}\frac{d}{dt}\vec{x}(t) = V^{-1}A\vec{x}(t) \tag{22}$$

$$\implies \frac{d}{dt}V^{-1}\vec{x}(t) = \Lambda\vec{z}(t) \tag{23}$$

$$\implies \frac{d}{dt}\vec{z}(t) = \begin{bmatrix} -6 & 0 \\ 0 & -1 \end{bmatrix} \vec{z}(t) \tag{24}$$

Because differentiation is linear, we can go from (22) to (23). In equation (24), we have successfully uncoupled our equations and we can proceed to solve them independently as mentioned earlier:

$$\frac{d}{dt}z_1(t) = -6z_1(t) \implies z_1(t) = k_1 e^{-6t}$$
$$\frac{d}{dt}z_2(t) = -z_2(t) \implies z_2(t) = k_2 e^{-t}$$

Next, we need to solve for our constants k_1 and k_2. Recall our initial conditions, $v_1(0) = v_2(0) = 1V$. Hence, $z_1(0)$ and $z_2(0)$ are given by:

$$\vec{z}(0) = V^{-1}\begin{bmatrix} v_1(0) \\ v_2(0) \end{bmatrix} \implies \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{3}{\sqrt{5}} \end{bmatrix} \tag{25}$$

Therefore, $k_1 = -\frac{1}{\sqrt{5}}$ and $k_2 = \frac{3}{\sqrt{5}}$ meaning we have solutions for $z_1(t)$ and $z_2(t)$. Our final step is to transform back into our original variable $\vec{x}$ as follows to find $v_1(t)$ and $v_2(t)$:

$$\vec{x} = V\vec{z} \tag{26}$$

$$= \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{5}}e^{-6t} \\ \frac{3}{\sqrt{5}}e^{-t} \end{bmatrix} \tag{27}$$

$$= \begin{bmatrix} \frac{2}{5}e^{-6t} + \frac{3}{5}e^{-t} \\ -\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t} \end{bmatrix} \tag{28}$$

For $t \geq 0$, we find that $v_1(t) = \frac{2}{5}e^{-6t} + \frac{3}{5}e^{-t}$ and $v_2(t) = -\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t}$. Figure 3 is a plot of our solutions:

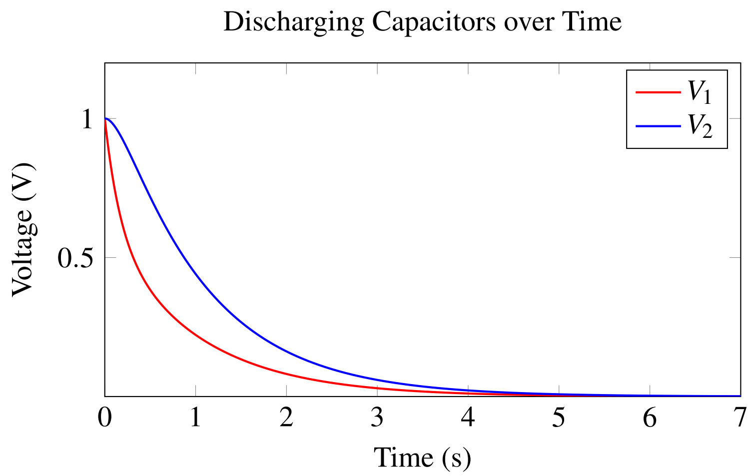


Figure 3: Initial Conditions: $v_1(0) = 1V$ and $v_2(0) = 1V$

6 Nonhomogenous Systems

Now that we have a good understanding of the homogenous case, let's look at the voltage transients of charging our two capacitor system. In this case, we have two uncharged capacitors, i.e. $v_1(0) = v_2(0) = 0\,\text{V}$, and we apply a voltage $V_{in} = 1V$ at time $t > 0$. We get the following branch equations:

$$v_2 = v_1 - I_2 R_2 \qquad I_2 = C_2 \frac{d}{dt} v_2 \tag{29}$$

$$V_{in} - v_1 = I_1 R_1 \Rightarrow I_1 = \frac{V_{in} - v_1}{R_1} \tag{30}$$

$$I_1 = I_2 + C_1 \frac{d}{dt} v_1 \tag{31}$$

Hence, our matrix differential equation is:

$$\frac{d}{dt}\vec{x}(t) = \frac{d}{dt}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right) & \frac{1}{R_2C_1} \\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} \frac{V_{in}}{R_1C_1} \\ 0 \end{bmatrix} \tag{32}$$

$$= \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} = A\vec{x} + \vec{b}. \tag{33}$$

Looking back at our diagonalization process, from (20) we can define a new variable $\vec{z}$ to get the differential equation

$$\frac{d}{dt}\vec{z}(t) = \Lambda\vec{z}(t) + \vec{c} \qquad \vec{c} = V^{-1}\vec{b} = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{6}{\sqrt{5}} \\ \frac{3}{\sqrt{5}} \end{bmatrix} \tag{34}$$

where $\vec{c} = V^{-1}\vec{b}$. We evaluate $\vec{c}$ and uncouple our system of differential equations and as a result, we get two first order scalar differential equations with a constant input.

Since the initial condition $\vec{x}(0) = \vec{0}$, the initial condition $\vec{z}(0) = V^{-1}\vec{x}(0) = \vec{0}$.

$$\frac{d}{dt}z_1(t) = -6z_1(t) - \frac{6}{\sqrt{5}} \implies z_1(t) = \frac{1}{\sqrt{5}}(e^{-6t} - 1) \tag{35}$$

$$\frac{d}{dt}z_2(t) = -z_2(t) + \frac{3}{\sqrt{5}} \implies z_2(t) = -\frac{3}{\sqrt{5}}(e^{-t} - 1) \tag{36}$$

Lastly, we convert our solution $\vec{z}$ back to $\vec{x}$.

$$\vec{x}(t) = V\vec{z}(t) \tag{37}$$

$$= \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}}(e^{-6t} - 1) \\ -\frac{3}{\sqrt{5}}(e^{-t} - 1) \end{bmatrix} \tag{38}$$

$$= \begin{bmatrix} 1 - \frac{2}{5}e^{-6t} - \frac{3}{5}e^{-t} \\ 1 + \frac{1}{5}e^{-6t} - \frac{6}{5}e^{-t} \end{bmatrix} \tag{39}$$

Finally, we have $v_1(t) = 1 - \frac{2}{5}e^{-6t} - \frac{3}{5}e^{-t}$ and $1 + \frac{1}{5}e^{-6t} - \frac{6}{5}e^{-t}$. Figure 4 is a plot of our solutions.

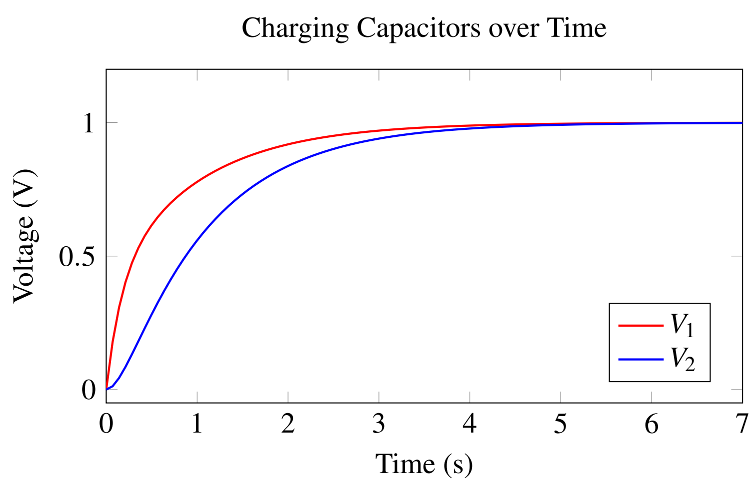


Figure 4: Voltage transients for charging capacitors

7 The Change of Basis Perspective

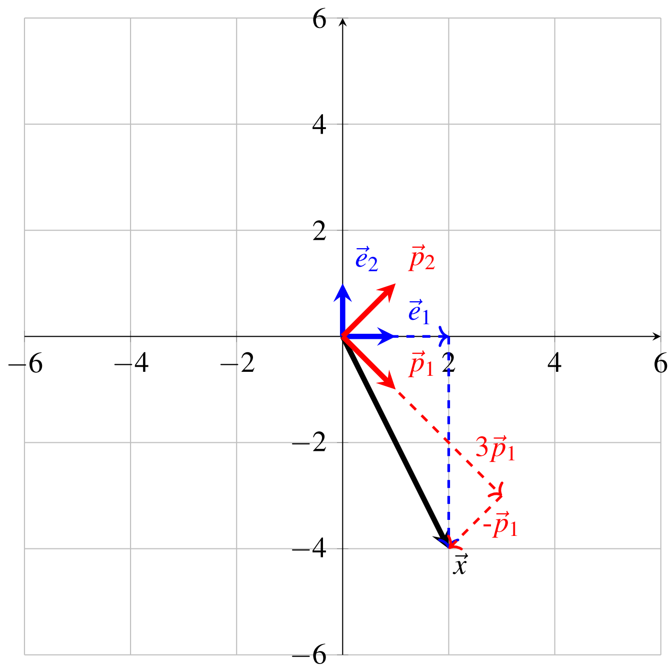
In this section of the note, we go back to our diagonalization process and draw the connection between eigendecomposition and coordinate systems. This provides a perspective where we are in fact transforming our coordinates into a basis in which the matrix A has a diagonal representation Λ.

Let's start with a vector $\vec{x} \in \mathbb{R}^n$. This vector represents a point in space. When you think about this vector written out using the coordinates, you are scaling the vectors in the standard basis (i.e. the columns of the identity matrix, I) by the components of $\vec{x}$ and then adding them up.

For example, we can write the vector $\vec{x} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ as the linear combination $\vec{x} = 2\vec{e}_1 - 4\vec{e}_2$ where $\vec{e}_1$ and $\vec{e}_2$ are the standard basis vectors. But, suppose that I think about this vector in terms of a different set of directions. More concretely, I define a new coordinate system:

$$\vec{p}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \vec{p}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{40}$$

Then to represent the vector $\vec{x} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$, we would need an alternate name for this vector. A quick computation can show that $\vec{z} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ achieves this since $\vec{x} = 3 \cdot \vec{p}_1 - 1 \cdot \vec{p}_2$. [2]



The vectors that define this coordinate system form a basis, i.e. n linearly independent vectors $\vec{p}_1, \ldots \vec{p}_n$ defined with respect to the standard basis.

[2] We can find $\vec{z}$ by taking the inverse of P and multiply it by $\vec{x}$. We wil explore why this is the case in the next page.

7.1 Changing Coordinates

Now, let's say I have a vector $\vec{z}$ which I am representing as $\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$ measuring with respect to my coordinate system. How can I translate this to the coordinates you are familiar with? Well, instead of scaling the vectors of the standard basis, we could scale the vectors defining my new basis.

Suppose that both of us were thinking of the same physical point in space and hence the vector $\vec{x}$ in your basis is:

$$\vec{x} = z_1\vec{p}_1 + \cdots + z_n\vec{p}_n \tag{41}$$

$$= \begin{bmatrix} | & & | \\ \vec{p}_1 & \cdots & \vec{p}_n \\ | & & | \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \tag{42}$$

$$= P\vec{z} \tag{43}$$

This also tells us that if we have a vector $\vec{x}$ written in the standard basis, we can transform it to its representation using P-basis vectors, $\vec{z}$, through the computation $\vec{z} = P^{-1}\vec{x}$.

7.2 Matrices in Different Bases

To transform a vector from my basis to your standard basis involves just a matrix multiplication. If this is the case, what would the matrix A look like in a different basis?

The matrix A performs a linear tranformation, $\vec{y} = A\vec{x}$. We would like to visualize this transformation in a different basis. In other words, we want to find the linear transformation D that performs the action $\vec{w} = D\vec{z}$.

To do this, we must first change $\vec{x}$ into the basis P. This is done by left multiplying to get $\vec{z} = P^{-1}\vec{x}$. Then, we can apply the transformation D to get $\vec{w} = DP^{-1}\vec{z}$. Finally, since our vector $\vec{w}$ is in a different basis, we must convert it back to the standard basis by multiplying by P to get $\vec{y} = P\vec{w} = PDP^{-1}\vec{x}$. This shows that $A = PDP^{-1}$ or $D = P^{-1}AP$.

We summarize our results in the diagram given in Figure 5 using the up and down arrows.

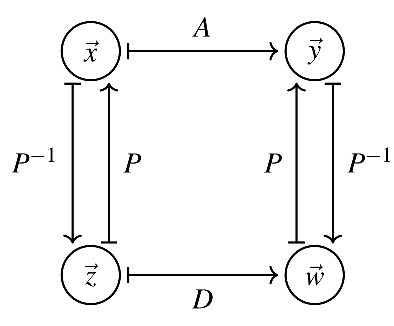


Figure 5: Change of Basis Mapping. It turns out $D = P^{-1}AP$ since matrix multiplication is done on the left.

7.3 Back to Diagonalization

Now that we've established what a matrix A looks like in a different basis, the question to ask is whether there is another basis within which this transformation is much simpler to understand.

Let's suppose we had our new basis V so that $\vec{x} = V\vec{z}$ and correspondingly, $\vec{z} = V^{-1}\vec{x}$. Similarly, $\vec{y} = V\vec{w}$ and correspondingly, $\vec{w} = V^{-1}\vec{y}$. Then:

$$A\vec{x} = AV\vec{z} = A(z_1\vec{v}_1 + z_2\vec{v}_2) \tag{44}$$
$$= z_1 A\vec{v}_1 + z_2 A\vec{v}_2 \tag{45}$$

Now, if chose our new basis to be the ones defined by the eigenvectors of A, then we can simplify:

$$= z_1\lambda_1\vec{v}_1 + z_2\lambda_2\vec{v}_2 \tag{46}$$
$$= \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{47}$$
$$= VD\vec{z} \tag{48}$$
$$= VDV^{-1}\vec{x} \tag{49}$$

where D is the diagonal matrix of eigenvalues and V is a matrix with the corresponding eigenvectors as its columns. Thus we have proved that $A = VDV^{-1}$. Furthermore, this also means that $D = V^{-1}AV$.

7.3.1 Repeated Eigenvalues

For a 2×2 matrix, it's possible that the two eigenvalues that you end up with have the same value, leading to a phenomenon called a **repeated eigenvalues**. This repeated eigenvalue can have one or two dimensional eigenspace (unlike a single, unrepeated eigenvalue, which will only have a one dimensional eigenspace).

For example, the following matrix has a repeated eigenvalue of λ.

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

The λ-**eigenspace** of this matrix is all of $\mathbb{R}^2$ since for any vector $\vec{v} \in \mathbb{R}^2$, $A\vec{v} = \lambda\vec{v}$.

We can also have examples like

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

that have a single eigenvalue $\lambda = 0$. (Easy to see by looking at the characteristic equation $\lambda^2 = 0$.) In this case, the relevant eigenspace is one-dimensional — only $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and its multiples are eigenvectors here.

8 Defective Matrices

In our approach to solve a system of differential equations, we developed a methodology in which we could turn a system of differential equations into n first-order scalar differential equations. This methodology involved a process called **diagonalization** in which we viewed a matrix A through its representation in a basis made up of eigenvectors. As a result, the matrix A had a diagonal representation Λ in our new basis.

However, note that each time we performed this process, we assumed that A was diagonalizable, or has a full basis consisting of eignevectors. Sadly, not every matrix has n linearly independent eigenvectors as we see above. In the next note, we will look at this case and a physical phenomena that arises from it. But for the time being, the best we can do is hope that our matrix A is diagonalizable.

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