EECS 16B Designing Information Devices and Systems II
Fall 2020 UC Berkeley

Note 6

Introduction

In the past couple of notes, we've developed a methodology to analyze the time behavior of circuits using differential equations. We started with scalar differential equations and were able to build it up into higher order differential equations. For a circuit with a large number of resistors, capacitors, and inductors, this analysis can get very difficult when analyzing arbitrary input signals.

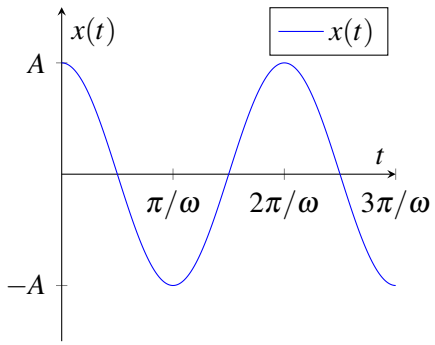
However, if we were to restrict our inputs to be **sinusoidal** currents and voltages, we will be able to create a new procedure very similar to Nodal Analysis from EE16A. This is sometimes called **frequency analysis** and it focuses on analyzing the steady-state behavior of circuits with sinusoidal inputs.

Sinusoidal inputs are very commonly used in the real world through contexts such as audio and WiFi. Understanding sinusoidal behavior will let us perform very powerful tasks such as **filtering** noise or designing power transformers. One of the key ideas in this note will be to rephrase our problem in terms of *frequency* instead of *time.*

1 Sinusoids

Before diving into any circuit analysis, let us define what it means to be **sinusoidal**. We say a function $x(t)$ is sinusoidal if it can be represented as $x(t) = A\cos(\omega t + \phi)$.[1]

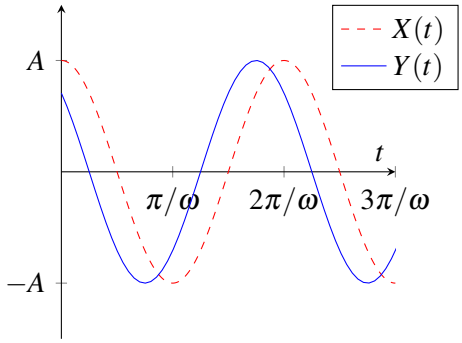
Let's start by plotting the function$x(t) = A\cos(\omega t)$



There are a couple properties of $x(t)$ that are immediately apparent from the figure. We call the maximum value of x above the mean the **amplitude** (A), and the spacing between repetitions of the function the **period** ($T = 2\pi/\omega$).

[1] In EE16B we will always use cosine references when talking about sinusoids. Note that sines and cosines are both "sinusoidal" since $\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$.

However, there's one other important property of sinusoids: their **phase**. Consider the function $Y(t) = A\cos(\omega t + \phi)$.



Here, ϕ represents the **phase shift** of Y with respect to X. As can be seen, a positive phase shift moves the function to the left by that amount. In particular, notice that the sine and cosine functions are really the same sinusoid, with each just the other after a $\pi/2$ radian phase shift in the appropriate direction.

2 Scalar Linear First-Order Differential Equations

Now going back to differential equations, let us recall what happens when we provide an exponential input e^{st}. For simplicity, consider the differential equation

$$\frac{d}{dt}x(t) = \lambda x(t) + u(t);\ x(0) = x_0$$

where the input $u(t)$ is of the form $u(t) = ke^{st}$ where $s \neq \lambda$.

From Note 3, we saw that the solution to this differential equation is of the form

$$x(t) = \left(x_0 - \frac{k}{s-\lambda}\right)e^{\lambda t} + \frac{k}{s-\lambda}e^{st},$$

What is interesting about this solution is that if $\lambda < 0$ then the steady state of $x(t)$ will be a scalar multiple of $u(t)$. Again, recall from Note 3 that the steady state only involved the e^{st} term.

$$x_{ss}(t) = \frac{k}{s-\lambda}e^{st} \tag{1}$$

But what about for complex λ? We can try writing a complex λ in the form $\lambda = \sigma + j\omega$, we see that

$$e^{\lambda t} = e^{(\sigma + j\omega)t} = e^{\sigma t}e^{j\omega t}. \tag{2}$$

To further simplify this, we can apply Euler's formula and see that

$$\left|e^{\lambda t}\right| = \left|e^{\sigma t}(\cos(\omega t) + j\sin(\omega t))\right| = \left|e^{\sigma t}\right|$$

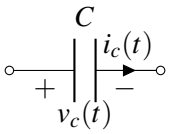
The first term in the product is a real exponential, which we know decays to zero exactly when $\mathrm{Re}[\lambda] = \sigma < 0$. The second term is a sum of two sinusoids that will always have a combined magnitude of 1.

To summarize, the homogeneous response $e^{\lambda t}$ will decay to zero whenever $\text{Re}[\lambda] < 0$. In the context of circuits with resistors, capacitors, and inductors, we almost always have this condition satisfied. This means for an exponential input of the form e^{st}, the steady state solution of the circuit will also be of the form e^{st}.

3 Circuits with Exponential Inputs

For a large circuit involving resistors, capacitors, and inductors, if we tried to solve the circuit using differential equations, every capacitor and inductor adds an extra derivative to the system making it difficult to solve. However, let's take a look at what happens in the special case that our circuit was driven by an exponential input e^{st} for some constant s.

Consider a capacitor C within the circuit with a voltage different $v_c(t)$ acrosss it and a current $i_c(t)$ flowing through it:



If the circuit were driven by an exponential function e^{st}, then at steady state, we know from our understanding of differential equations that $v_c(t) = \widetilde{V}e^{st}$ and $i_c = \widetilde{I}e^{st}$ where $\widetilde{V}$ and $\widetilde{I}$ are some arbitrary scalars.

The voltage-current relation of a capacitor tells us that

$$i_c(t) = \widetilde{I}e^{st} = C\frac{d}{dt}v_c(t) = C\frac{d}{dt}\widetilde{V}e^{st} = sC\widetilde{V}e^{st} \implies \widetilde{I} = sC \cdot \widetilde{V}$$

Critically, note that the relation between $\widetilde{I}$ and $\widetilde{V}$ resembles that of Ohm's law and has no time-dependence - it is a purely linear equation.

Similar equations can be obtained (this is a useful exercise to do) for an inductor $\widetilde{V} = \widetilde{I} \cdot sL$ and a resistor $\widetilde{V} = \widetilde{I} \cdot R$. Rewriting the capacitor relationship to be in the same form, we see $\widetilde{V} = \widetilde{I} \cdot \frac{1}{sC}$.

This suggests that we can view capacitors, inductors, and resistors as all being similar. In effect, capacitors and inductors just have s-dependent resistances that we will call s-impedances. The term **impedance** is a generalization of resistance and is defined through the voltage-current ratio

$$Z = \frac{\widetilde{V}}{\widetilde{I}} \tag{3}$$

In 16A, when we did all of our analysis with resistors, no differential equations arose. This was because of Ohm's Law $V = IR$. Using our new idea of impedances, we are going to extend Ohm's Law to capacitors and inductors so we can use all of our circuit analysis techniques from 16A.

3.1 Defining the Phasor

We were able to define s-impedances for any input of the form $u(t) = e^{st}$. However, how does this all relate to our main goal of sinusoidal analysis? It turns out that sinusoids are a special case of exponential functions!

Now, in our circuit differential equations, our input $u(t)$ was a sinusoid of the form

$$u(t) = A\cos(\omega t + \phi)$$

Let's see how we can rewrite them in terms of exponential functions. To do this, we can combine Euler's formula with the properties of complex conjugates to determine that

$$e^{j\theta} + e^{-j\theta} = (\cos\theta + j\sin\theta) + (\cos\theta - j\sin\theta) = 2\cos\theta.$$

In other words, starting with two complex exponentials, we have pulled out a purely real sinusoid!

Now let's see what happens if we use the same algebraic manipulations to express an arbitrary sinusoidal voltage $v(t) = V_0\cos(\omega t + \phi)$ in terms of exponential functions.

$$v(t) = V_0\cos(\omega t + \phi) = \frac{1}{2}V_0 e^{j(\omega t+\phi)} + \frac{1}{2}V_0 e^{-j(\omega t+\phi)}$$

Therefore, we can express an arbitrary sinusoid as a linear combination of two exponential functions! Notice that the coefficients of the two exponential functions are complex conjugates of one another. Thus, we can rewrite the above as[2]

$$v(t) = \frac{1}{2}V_0 e^{j\phi}e^{j\omega t} + \frac{1}{2}\overline{V_0 e^{j\phi}}e^{-j\omega t}.$$

The $V_0 e^{j\phi}$ term resembles the $\widetilde{V}$ term that we denoted as an arbitrary scalar in the previous section. We will call this coefficient the **phasor** $\widetilde{V}$ representing $v(t)$ and denote it as [3]

$$\widetilde{V} = V_0 e^{j\phi} \tag{4}$$

The phasor is a complex scalar that is a different way to represent our sinusoid **without using time.** Based on this definition of phasors, we see that our voltage $v(t)$ can be written in the form

$$v(t) = \frac{1}{2}\widetilde{V}e^{j\omega t} + \frac{1}{2}\overline{\widetilde{V}}e^{-j\omega t} \tag{5}$$

3.1.1 Linear Circuits Don't Change Frequency

With a phasor, notice how we are representing a sinusoid by its amplitude and phase, but not frequency. We are allowed to do this and choose to do so since linear circuits (resistors, capacitors, and inductors) will never alter the frequency of a sinusoid.

Concept Check: Why does a linear circuit never alter the frequency of a sinusoid?
Solution: It comes from the fact that e^{st} is an eigenfunction of the derivative operator. For $s = j\omega$, we will always have an $e^{j\omega t}$ term and its conjugate $e^{-j\omega t}$ forming our sinusoid of frequency ω.

Based on this fact, in the next section, we look further into the definition of this phasor and how it relates to our definition of s-impedances.

[2] Remember that $e^{-j\phi} = \overline{e^{j\phi}}$

[3] We use capital letters for phasors to distinguish them from time domain signals.

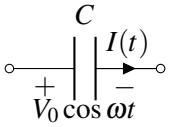
4 Phasors

In principle, at this point we already know what to do when given a circuit with sinusoidal inputs all at the same frequency. But it can be helpful to make sure that you understand the derivations.

We defined the phasor for a sinusoidal voltage $v(t) = V_0 \cos(\omega t + \phi)$ as $\widetilde{V} = V_0 e^{j\phi}$ and saw that linear circuits will not change the frequency of a sinusoid. A linear circuit component will only affect the amplitude and phase of a sinusoid so we deemed it sufficient to represent a sinusoid through this phasor representation.

While all the derivations can be done by writing $\cos(\theta)$ as a sum of complex exponentials, we can also say that $\cos(\theta) = \mathfrak{Re}[e^{j\theta}]$ to simplify our calculations.

Let's look at a capacitor provided with the sinusoidal voltage $v(t) = V_0 \cos(\omega t + \phi) = \mathfrak{Re}[\widetilde{V} e^{j\omega t}]$, as shown:



Now, by the capacitor current-voltage relationship, we know that

$$i(t) = C\frac{d}{dt}v(t) = C\frac{d}{dt}\mathfrak{Re}[\widetilde{V}e^{j\omega t}] \tag{6}$$

$$= C \cdot \mathfrak{Re}[\widetilde{V}\frac{d}{dt}e^{j\omega t}] = \mathfrak{Re}[j\omega C\widetilde{V}e^{j\omega t}] \tag{7}$$

We see that the current is indeed still a sinusoid of frequency ω, but the amplitude and phase have changed.

Now if we were to represent the current in terms of its phasor representation,

$$i(t) = I_0 \cos(\omega t + \theta) = \mathfrak{Re}[\widetilde{I}e^{j\omega t}] \tag{8}$$

Looking back at Equation (7), it follows that $\widetilde{I} = (j\omega C)\widetilde{V}$.

In other words, having already shown that all steady state circuit quantities will be sinusoids with frequency ω, we now in fact can relate the phasors of the voltage across and the current through a capacitor by a ratio that depends only on the frequency and the capacitance.

This is exactly the same as the s-impedance story we told earlier. Because of this, when dealing with sinusoidal inputs at frequency ω, we use $s = +j\omega$ and just call the s-impedance, the *impedance*. The $+j\omega$ is understood from context.

As before, this can be thought of as the "resistance" of a capacitor, since it relates the phasor representations voltage and current over and through the element by a constant ratio. For a capacitor, the impedance is

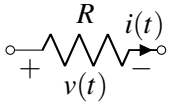
$$Z_C = \frac{\widetilde{V}}{\widetilde{I}} = \frac{1}{j\omega C}.$$

The interesting fact is that the impedance for the capacitor is imaginary, but more on that later.

We will now quickly perform a similar analysis for inductors and resistors on the next page.

Impedance of a Resistor

Imagine some resistor R as follows:



Let $i(t)$ be represented by some phasor $\widetilde{I}$. Thus, by Ohm's Law,

$$v(t) = i(t) \cdot R = R \cdot \mathfrak{Re}[\widetilde{I}e^{j\omega t}] \tag{9}$$

$$= \mathfrak{Re}\left[\widetilde{I} \cdot Re^{j\omega t}\right] \tag{10}$$

so we may represent the output voltage with the phasor

$$\widetilde{V} = \widetilde{I} \cdot R,$$

so the impedance is clearly

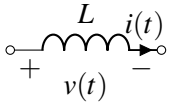
$$Z_R = R.$$

From this, we see that the impedance behaves very much like the resistance does, except that it generalizes to other circuit components as well.

Impedance of an Inductor

Now, we will consider inductors. We've seen that any sinusoidal function can be represented by a phasor. Since we know our steady state will a sinusoid with frequency ω, we start with a sinusoidal current and work in the opposite direction to calculate the impedance of an inductor.

Consider an inductor with voltage and current across it as follows:



Let $i(t)$ be represented by some phasor $\widetilde{I}$. The voltage-current relationship of an inductor tells us that

$$v(t) = L\frac{d}{dt}i(t) = L\frac{d}{dt}\mathfrak{Re}[\widetilde{I}e^{j\omega t}] \tag{11}$$

$$= L \cdot \mathfrak{Re}[\widetilde{I}\frac{d}{dt}e^{j\omega t}] = \mathfrak{Re}[j\omega L\widetilde{I}e^{j\omega t}] \tag{12}$$

so the voltage can be represented by the phasor

$$\widetilde{V} = j\omega L\widetilde{I}.$$

Thus, the impedance of an inductor is

$$Z_L = j\omega L.$$

5 Circuit Analysis

At this point, observe that we have essentially obtained "equivalents" to Ohm's Law for inductors and capacitors, using the impedance to relate their voltage and current phasors.

5.1 The Phasor Transform

What allowed us to do this was the underlying assumption that our voltages and currents were all sinusoidal. Given a sinusoidal voltage or current $u(t)$ we could represent it as a phasor.

$$u(t) = A\cos(\omega t + \phi) \implies \widetilde{U} = Ae^{j\phi} \tag{13}$$

This is commonly referred to as the **Phasor Transform**. Given this phasor $\widetilde{U}$, we could then solve for all of our voltages and currents in our circuit in phasor form. But how would we convert our phasors back into sinusoids as a function of time?

Remember that phasors are a representation of sinusoids and are just a complex scalar. Therefore, we can define an **Inverse Phasor Transform** of the form

$$\widetilde{W} = Be^{j\theta} \implies w(t) = B\cos(\omega t + \theta) \tag{14}$$

5.2 How to do Circuit Analysis

How do we use our phasors? We provide a short guideline below on how to analyze a circuit using phasors.

1. Verify that the voltages and currents are indeed sinusoidal.
    - **Note: Phasor analysis only works on sinusoidal inputs!**
2. Transform all voltages and currents to the "Phasor Domain."
    - Phasor Domain a world in which time does not exist and everything is a phasor.
3. Solve for all voltages and currents.
    - You can use any KCL, KVL, Voltage Dividers, etc.
4. Bring all voltages and currents back to the "Time Domain" through the Inverse Transform.

5.3 The Evil Twin

Now the last remark we make about sinusoids $v(t) = V_0\cos(\omega t + \phi)$, we defined the phasor as $\widetilde{V} = V_0e^{j\phi}$ equipped to the exponential function $e^{j\omega t}$. However, we also had the "evil twin" conjugate phasor and its equipped exponential $e^{-j\omega t}$. So we might ask why the conjugate phasor isn't seen in our transform.

This is because the two phasors $\widetilde{V}$ and $\overline{\widetilde{V}}$ are related by conjugation. If we were to alter the phasor $\widetilde{V}$, we would also be altering $\overline{\widetilde{V}}$. So then you might ask, what is the purpose of this conjugate phasor? The conjugate phasor and its equipped exponential $e^{-j\omega t}$ exist and act as a shadow to $\widetilde{V}$ and $e^{j\omega t}$ to make our sinusoid a real function. Note that the function $e^{j\omega t}$ has a real and imaginary part and hence, its imaginary part can only be cancelled out by adding its conjugate.

5.4 KCL with Phasors

We will now try to show that a sum of sinusoidal functions is zero if and only if the sum of the phasors of each of those functions equals zero as well, to obtain a sort of "phasor-version" of KCL. Consider the sinusoids represented by the phasors: $\widetilde{I}_1, \widetilde{I}_2, \ldots, \widetilde{I}_n$.

Let $I_k(t)$ be the sinusoid represented by the phasor $\widetilde{I}_k$.

Observe that

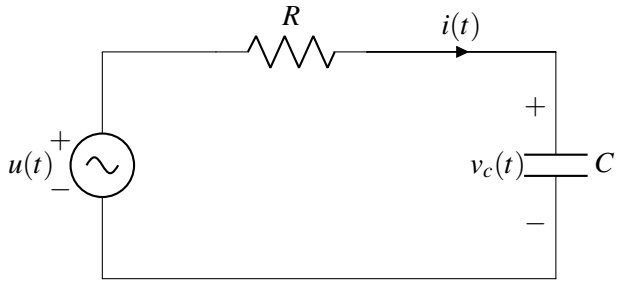
$$\begin{aligned}
& & \widetilde{I}_1 + \widetilde{I}_2 + \ldots + \widetilde{I}_n &= 0 \\
\iff & & \frac{1}{2}(\widetilde{I}_1 + \widetilde{I}_2 + \ldots + \widetilde{I}_n)e^{j\omega t} &= 0 \\
\iff & \frac{1}{2}(\widetilde{I}_1 + \widetilde{I}_2 + \ldots + \widetilde{I}_n)e^{j\omega t} + \frac{1}{2}\overline{(\widetilde{I}_1 + \widetilde{I}_2 + \ldots + \widetilde{I}_n)e^{-j\omega t}} &= 0 \\
\iff & & \sum_{k=1}^{n} \frac{1}{2}\left(\widetilde{I}_k e^{j\omega t} + \overline{\widetilde{I}_k e^{-j\omega t}}\right) &= 0 \\
\iff & & I_1(t) + I_2(t) + \ldots + I_n(t) &= 0,
\end{aligned}$$

so we have proved that a sum of sinusoids is zero if and only if the sum of their corresponding phasors is zero as well. This result can be thought of as a generalization of KCL to phasors.

5.5 Circuit Example

Putting everything together, we have now successfully generalized all of our techniques of DC analysis to frequency analysis. We can finally consider some basic circuits, to verify that our technique works correctly.

Consider a voltage divider, where instead of one resistor we introduce a capacitor, as follows:



Let $u(t) = V_0 \cos\left(\omega t + \frac{\pi}{2}\right)$. We are interested in finding how the voltage $v_c(t)$ varies over time. Note that it is possible to solve this problem using differential equations, but we will now take advantage of phasors.[4]

Recall that we proved the voltage divider equation in the context of DC circuit analysis. However, that proof carries over to the phasor domain in a straightforward manner. Thus, the phasor $\widetilde{V}_c$ representing the voltage $v_c(t)$ can be represented in terms of the phasor $\widetilde{U}$ representing the supply voltage as follows:

$$\widetilde{V}_c = \frac{Z_C}{Z_C + Z_R}\widetilde{U},$$

where Z_C and Z_R are the impedances of the capacitor and resistor, respectively. Note also that, since the supply is at frequency ω, all other voltages and currents in the system will also be at the same frequency ω.

[4] We in fact try this in the Input Note, but recall how tedious the integral becomes.

Thus, using our results from earlier, we know that

$$Z_C = \frac{1}{j\omega C},\ Z_R = R$$

Substituting these values into our equation for $\widetilde{V}_c$, we find that

$$\widetilde{V}_c = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C}+R} V_0 e^{j\frac{\pi}{2}} = V_0 \frac{j}{1+j\omega RC} \tag{15}$$

$$= \frac{V_0}{\sqrt{1^2+(\omega RC)^2}} e^{j(\frac{\pi}{2}-\text{atan2}(1,\omega RC))} \tag{16}$$

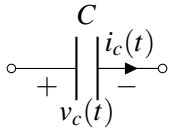
Note that we have converted $\frac{j}{1+j\omega RC}$ into polar form.[5] This would be much easier to write out if we were given component values and a frequency ω.

Lastly, we can convert this phasor back into the time-domain to obtain

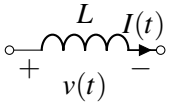
$$v_c(t) = \frac{V_0}{\sqrt{1^2+(\omega RC)^2}} \cos\left(\omega t + \frac{\pi}{2} - \text{atan2}(1,\omega RC)\right) \tag{17}$$

5.6 Phasors at DC

We've mentioned multiple times throughout this note that Phasor Analysis only works when the inputs are sinusoidal. However, since $1 = e^{j\cdot 0}$, what would happen if we did Phasor Analysis on *DC* voltages and currents? This fact seems to imply that DC voltages are "sinusoidal" with zero frequency so let us look at the impedances of a capacitor and inductor for $\omega = 0$.



Since the impedance of a capacitor is $Z_c = \frac{1}{j\omega C}$, we see that a capacitor would have infinite impedance and act as an open circuit. Similarly looking at an inductor at DC,



The impedance of the inductor would be $Z_L = 0$ meaning an inductor would have zero impedance and acts as a short circuit.

Remember that when using phasors, we were looking at the steady state behvaior of our circuit. In addition, recall that for DC voltages, no current flows through the capacitor and inductors behave as shorts in steady state. Therefore, performing phasor analysis with $\omega = 0$ matches our intuition of what would happen with DC inputs at steady state.

[5] Remember that for a complex number $w = \frac{z_1}{z_2}$, the magnitude $|w| = \frac{|z_1|}{|z_2|}$ and the phase is $\angle w = \angle z_1 - \angle z_2$.

Contributors:

- Rahul Arya.
- Anant Sahai.
- Jaijeet Roychowdhury.
- Taejin Hwang.

