EECS 16B Designing Information Devices and Systems II
Fall 2020 UC Berkeley

Note 8

Introduction

In the last note, we developed fundamental techniques to design first and second-order **filters** by computing a **transfer function** $H(\omega)$ and analyzing its behavior for various values of ω. To get a better understanding of our circuits, we decided to plot the transfer function over various frequencies using numerical tools such as Matlab or Python.

In this note, we will develop a methodology to approximate the behavior of our transfer function through **Bode plots.** We will continue to emphasize the importance of **approximating** the behavior of our circuits. Often times instead of trying to understanding the exact, precise behavior of a system, we look at an approximation that is equally valid and gives us the perfect amount of information to make our design choices.

1 Rational Transfer Functions

When we write out the transfer function of an arbitrary circuit, it can always be expressed as a **rational transfer function** of the following form.

$$H(\omega) = K\frac{N(\omega)}{D(\omega)} = K\frac{(j\omega)^{N_{z0}}}{(j\omega)^{N_{p0}}}\left(\frac{(j\omega)^n + \alpha_{n-1}(j\omega)^{n-1} + \ldots + \alpha_1(j\omega) + \alpha_0}{(j\omega)^m + \beta_{m-1}(j\omega)^{m-1} + \ldots + \beta_1(j\omega) + \beta_0}\right) \tag{1}$$

$$= K\frac{(j\omega)^{N_{z0}}\left(1 + j\frac{\omega}{\omega_{z_1}}\right)\left(1 + j\frac{\omega}{\omega_{z_2}}\right)\cdots\left(1 + j\frac{\omega}{\omega_{zn}}\right)}{(j\omega)^{N_{p0}}\left(1 + j\frac{\omega}{\omega_{p_1}}\right)\left(1 + j\frac{\omega}{\omega_{p_2}}\right)\cdots\left(1 + j\frac{\omega}{\omega_{pm}}\right)} \tag{2}$$

This follows from the Fundamental Theorem of Algebra which states that a degree n polynomial has n complex roots. This rational form will help us tremendously when plotting.

1.1 Poles and Zeros

In the rational form shown above, we define constants ω_z as **zeros** and ω_p as **poles.** Note that in standard literature, zeros are defined to be the roots of $N(\omega)$ while poles are the roots of $D(\omega)$.[1] In addition to this, we refer to the $(j\omega)$ term as a **zero at the origin** while $\frac{1}{j\omega}$ is a **pole at the origin**.

Note that **pole** and **zero** frequencies are a generalization of cutoff frequencies. Often times, $\left|H(\omega)\right|$ at a pole or zero will not be equal to $\frac{1}{\sqrt{2}}$. However, they are of utmost importance since they represent a point of the transfer function where the behavior begins to change qualitatively. For this reason, you may see the term **break frequency** in other literature which refers to the idea that the transfer function starts changing at pole or zero frequencies.

[1] Technically if $s = j\omega$, then the roots of $N(s)$ and $D(s)$ are $-\omega_z$ and $-\omega_p$. However, when plotting Bode plots, we refer to ω_z and ω_p as the zero and pole frequencies.

1.2 "Adding" Bode Plots

For two transfer functions $H_1(\omega)$ and $H_2(\omega)$, if $H(\omega) = H_1(\omega) \cdot H_2(\omega)$,

$$\log\left|H(\omega)\right| = \log\left|H_1(\omega) \cdot H_2(\omega)\right| = \log\left|H_1(\omega)\right| + \log\left|H_2(\omega)\right| \tag{3}$$

$$\angle H(\omega) = \angle(H_1(\omega) \cdot H_2(\omega)) = \angle H_1(\omega) + \angle H_2(\omega) \tag{4}$$

As a consequence, when plotting $\left|H(\omega)\right|$ on a log scale, we can simply plot $\left|H_1(\omega)\right|$ and $\left|H_2(\omega)\right|$ and add the two together. This implies that we will be able to add the slopes of each zero and pole to provide a complete plot. In the next section we provide a further analysis on the meaning of zeros and poles and the idea of adding slopes.

We must be careful, however, to note that in most of our plots, the x-axis does *not* correspond to 0, so we can't simply "stack" the two plots.

1.3 Decibels

We define the decibel as the following:

$$20\log_{10}(|H(\omega)|) = |H(\omega)| \text{ [dB]}$$

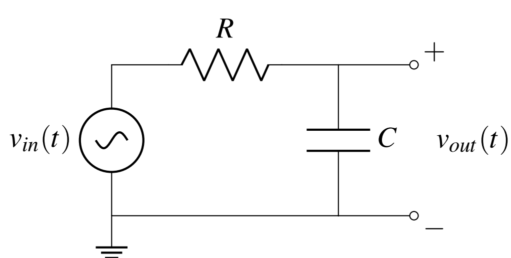
The origin of the decibel comes from looking at the ratio of the output and input power of the system.

$$\left|H(\omega)\right| \text{ [dB]} = 10\log\left|\frac{P_{out}}{P_{in}}\right| = 10\log\left|\frac{V_{out}}{V_{in}}\right|^2 = 20\log\left|\frac{V_{out}}{V_{in}}\right|$$

Therefore, 20 dB per decade is equivalent to one order of magnitude. **We won't be using dB when plotting, but understanding the conversion to dB will help when reading other resources on Bode plots.**

2 First Order Examples

Let us look back to the simplest example, a first-order RC low-pass filter with $R = 1\,\text{k}\Omega$ and $C = 1\,\text{nF}$.



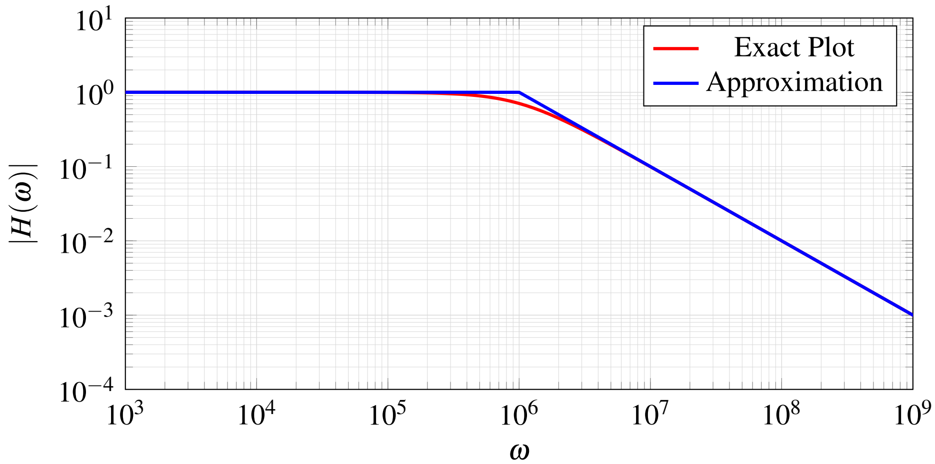
You might recall that this circuit has the following transfer function

$$H(\omega) = \frac{1}{1 + j\omega RC}$$

Notice how this transfer function is already in the rational form mentioned in the previous section and that is has a pole at $\omega_p = \frac{1}{RC}$. Now based on this information, let's take a look at various frequencies around $\omega_p = \frac{1}{RC} = 10^6$ and break it into cases:

- If $\omega \ll \omega_p$, then $\omega/\omega_p \approx 0$. Therefore $H(\omega) \approx 1$ which implies $\left|H(\omega)\right| \approx 1$.
- For $\omega = \omega_p$, then $H(\omega) = \frac{1}{1+j}$ meaning $\left|H(\omega)\right| = \frac{1}{\sqrt{2}}$.
- Lastly if, $\omega \gg \omega_p$, then $\omega/\omega_p \gg 1$. Therefore $H(\omega) \approx \frac{1}{j\omega/\omega_p} = -j\omega_p/\omega$ which implies $\left|H(\omega)\right| \approx \omega_p/\omega$. Therefore, if we plot $\left|H(\omega)\right|$ on a log scale, the magnitude will drop off with a slope of 1.[2]

Based on the analysis above, we can plot a **straight-line approximation** of $H(\omega)$ that is equal to 1 for $\omega < \omega_p$ and then drops off with a slope of 1 for $\omega > \omega_p$. We plot both the approximation and exact values of $\left|H(\omega)\right|$ below to see how precise our approximation is.



The Bode approximation matches the true plot very accurately. The largest approximation error in the straight-line plot occurs at the pole frequency $\omega_p = \frac{1}{RC}$. We approximate $\left|H(\omega)\right|$ as 1 when in reality, it is equal to $\frac{1}{\sqrt{2}}$.

2.1 Single Zero

Now let's take a look at another transfer function with a single zero at $\omega_z = 10^6$.

$$H(\omega) = 1 + j\omega/\omega_z \tag{5}$$

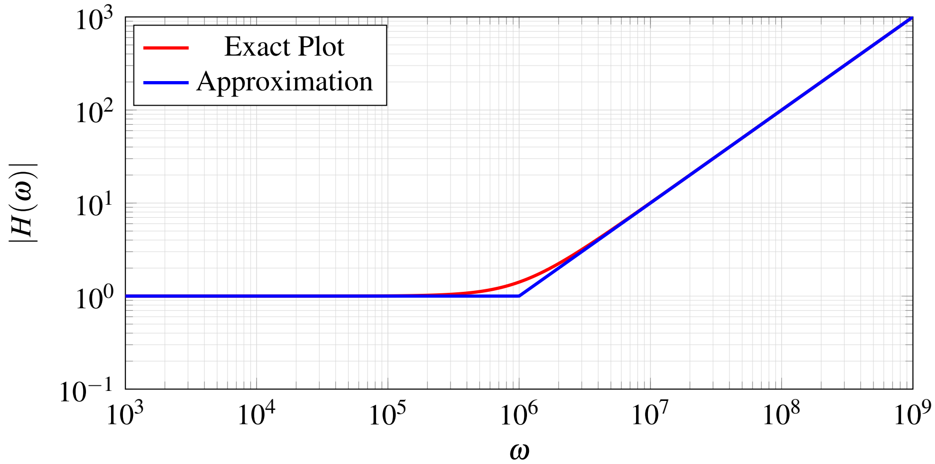
Note that it is impossible to implement this transfer function using a passive circuit since $\left|H(\omega)\right| \to \infty$ as $\omega \to \infty$. However, it is an important example as transfer functions can have a single zero and multiple poles to cancel out the effect of the zero.

Let's again analyze various frequncies around ω_z and how they affect the value of $\left|H(\omega)\right|$.

- If $\omega \ll \omega_z$, then $\omega/\omega_z \approx 0$. Therefore $H(\omega) \approx 1$ which implies $\left|H(\omega)\right| \approx 1$.
- For $\omega = \omega_z$, then $H(\omega) = 1 + j$ meaning $\left|H(\omega)\right| = \sqrt{2}$.
- Lastly if, $\omega \gg \omega_z$, then $\omega/\omega_z \gg 1$. Therefore $H(\omega) \approx j\omega/\omega_z$ which implies that $\left|H(\omega)\right| \approx \omega/\omega_z$. Therefore, we see that $\left|H(\omega)\right|$ increases with a slope of 1 on a log scale.

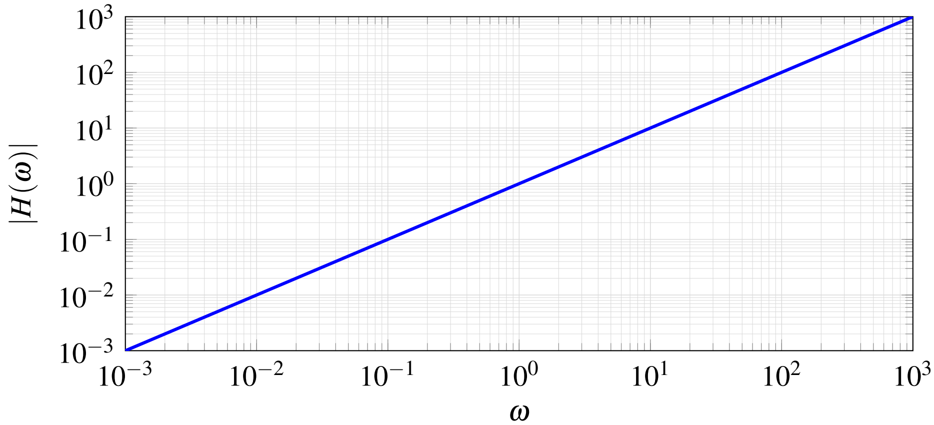
[2] If you aren't sure why this is the case, notice that $\log\left|H(\omega)\right| = \log\omega_p - \log(\omega)$. Since the line $y = mx + b$, has a slope of m, we can equate $y = \log\left|H(\omega)\right|$, $x = \log\omega$ and $b = \log\omega_p$.

Based on the analysis above, we can plot a **straight-line approximation** of $H(\omega)$ that is equal to 1 for $\omega < \omega_z$ and then increases with a slope of 1 for $\omega > \omega_z$. We again plot both the approximation and exact values of $|H(\omega)|$ to see how precise our approximation is.

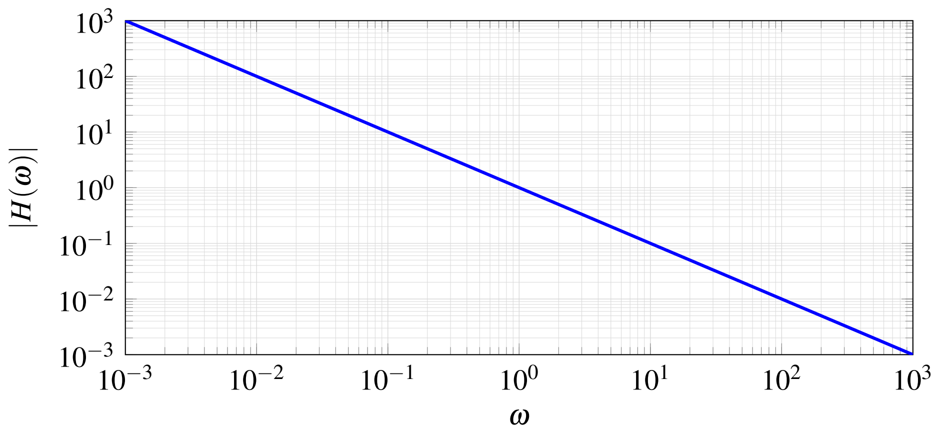


2.2 At the Origin

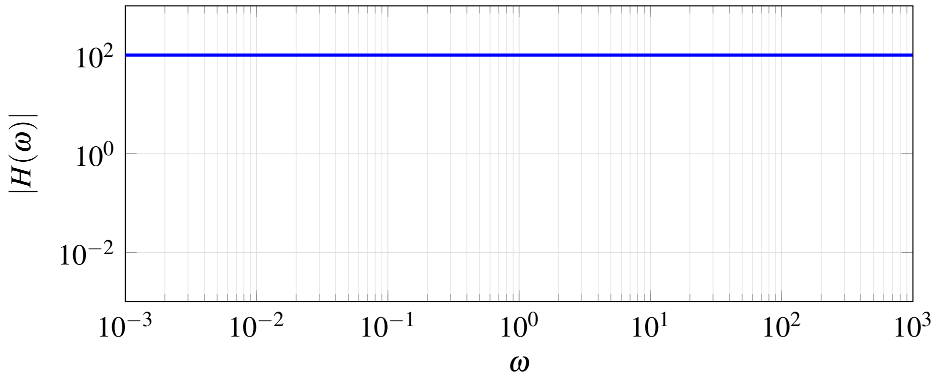
To plot a zero at the origin, notice that $H(\omega) = j\omega$ has magnitude ω and phase 90°. If our transfer function has a zero at the origin, it will start off with a slope of 1.



To plot a pole at the origin, notice that $H(\omega) = \frac{1}{j\omega}$ has magnitude ω and phase $-90°$. If our transfer function has a pole at the origin, it will start off with a slope of -1.



Lastly, we show the plot of a constant $K = 100$. As expected, the plot remains constant. This implies that multiplication by K will shift up the entire bode plot up by K.

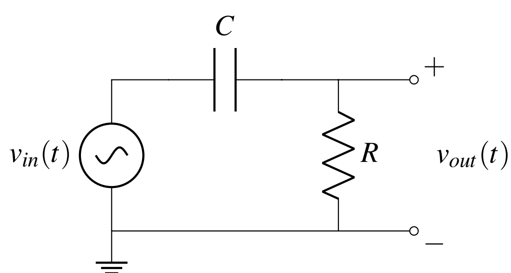


3 Higher-Order Filters

Now that we understand how to plot first-order Bode plots, we'll look at more complicated examples that involve multiple poles and zeros. Remember that we are able to treat a higher-order Bode plot as the product of multiple first-order Bode plots and "add" up our results together. The examples in this section should help illustrate this idea.

3.1 High-pass Filter

Let us take another look at the first-order RC high-pass filter with $R = 1\,\text{k}\Omega$ and $C = 1\,\text{nF}$.



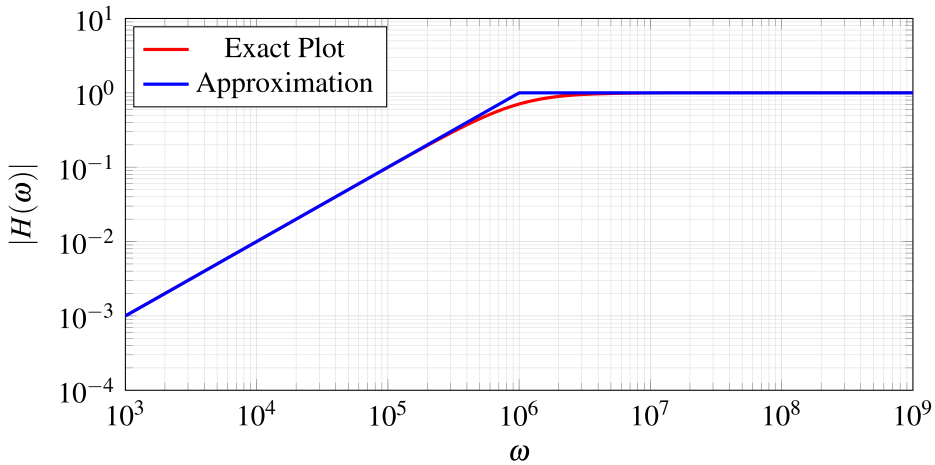
The transfer function of this circuit as you may recall is of the form

$$H(\omega) = \frac{j\omega RC}{1 + j\omega RC} \tag{6}$$

We can break this transfer function down into its rational form: A constant RC term, a single zero at the origin, and a pole at $\omega_p = \frac{1}{RC}$. Each individual component will contribute to the plot of $H(\omega)$ and we can add all of their effects together.

$$H(\omega) = RC \cdot (j\omega) \cdot \frac{j\omega RC}{1 + j\omega RC} \tag{7}$$

We first plot the magnitide Bode plot, then provide an analysis of each constituent component



To provide an analysis for this Bode plot, we note that there is a constant, single zero, and single pole in that exact order. The constant term $K = RC = 10^{-6}$ shifts the entire plot down by 10^6. Since there is a single zero at the origin, the plot must start with a slope of 1. Lastly, the pole at $\omega_p = 10^6$ will provide a slope of -1 that cancels out with the current slope of 1 from the zero. Therefore, for $\omega > \omega_p$, the net slope will be zero and the plot remains constant.

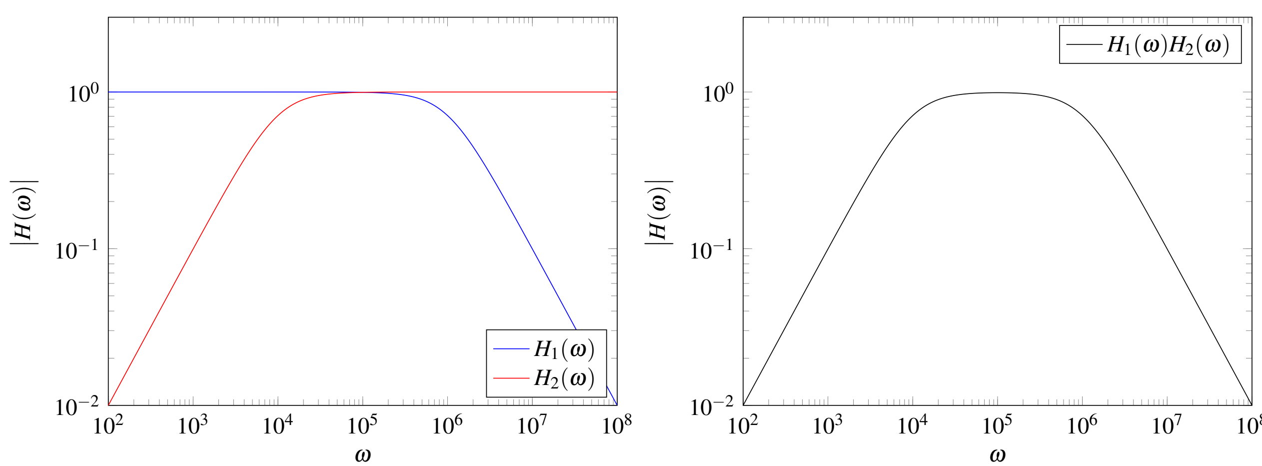
3.2 Band-pass Filter

Recall the band-pass filter we designed in the previous note by cascading a low-pass and a high-pass filter.

$$H(\omega) = H_{LP}(\omega) \cdot H_{HP}(\omega) = \frac{1}{1 + j\omega/10^6} \cdot \frac{j\omega/10^4}{1 + j\omega/10^4} \tag{8}$$

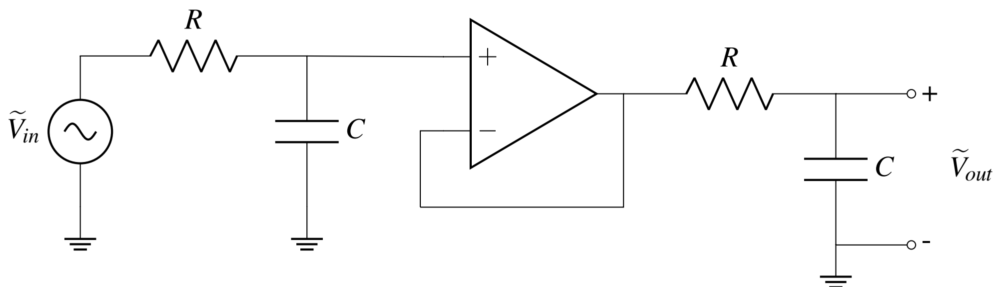
The cutoff frequency for the high-pass is 10^4 while the cutoff for the low-pass is 10^6.

Following this procedure of adding plots (with the individual filters on the left and the result on the right), we obtain



3.3 Low-Pass Filters

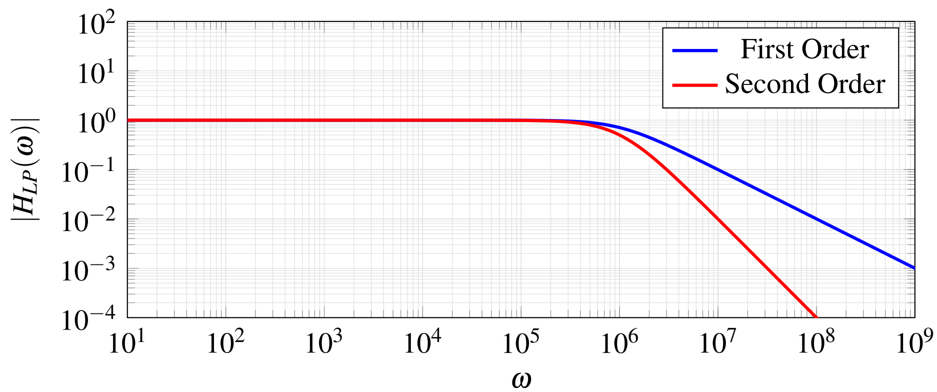
From our analysis of low-pass filters, we saw that the magnitude of $H(\omega)$ drops off by a factor of 10 for each decade of frequency after the cutoff ω_c. While this functions as a low-pass filter, in the ideal case, we would like to build a filter that drops off at a quicker rate after ω_c. Therefore, let's try cascading two low-pass filters of identical cutoff with a buffer in between.



We can compute the transfer function as

$$H_{LP}(\omega) = \frac{1}{(1 + j\omega RC)^2} \tag{9}$$

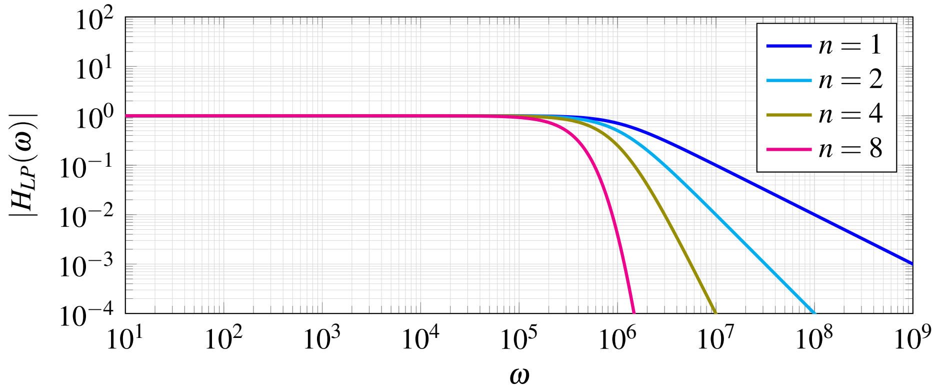
Plotting the magnitude of $H_{LP}(\omega)$, we see that $H(\omega)$ does indeed drop off at a quicker rate with slope 2 after the cutoff ω_c.



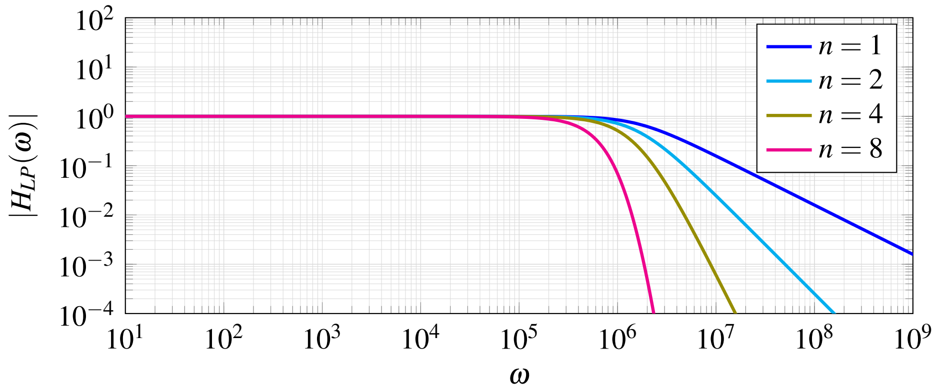
In fact, if we were to cascade even more low-pass filters, we approach an ideal low-pass filter in which

$$H(\omega) = \begin{cases} 1 & \omega < \omega_c \\ 0 & \omega \geq \omega_c \end{cases} \tag{10}$$

We show a plot of this effect below. For an n^{th} order filter, we see a dropoff of slope n after the cutoff. We will explore this effect in more detail in the next note.



When analyzing the Bode plot however, note that dropoff ocurrs before $\omega_p = \frac{1}{RC}$. This is because each time we cascade a low-pass filter, the magnitude drops off by a factor of $\frac{1}{\sqrt{2}}$ at ω_p. The Bode approximation is unable to capture this behavior. If we wanted to build something closer to the ideal low-pass filter, we need to shift $\frac{1}{RC}$ to be slightly greater than ω_c. We show a plot below where we set $\omega_p = \frac{1}{RC} = 10^{-6.2}$.



With this slight shift, we see a slight performance improvement, but it is quite expensive with all of the op-amps especially at the $n = 8$ case. There are an entire class of different filter designs each with its own tradeoffs. Some examples that you can look up are the **Butterworth**, **Lattice**, and **Sallen-Key** topologies.

4 More Examples

We provide more examples of transfer functions and their Bode plots to reinforce the idea of "adding" plots and slopes together.

4.1 Transfer Function Example

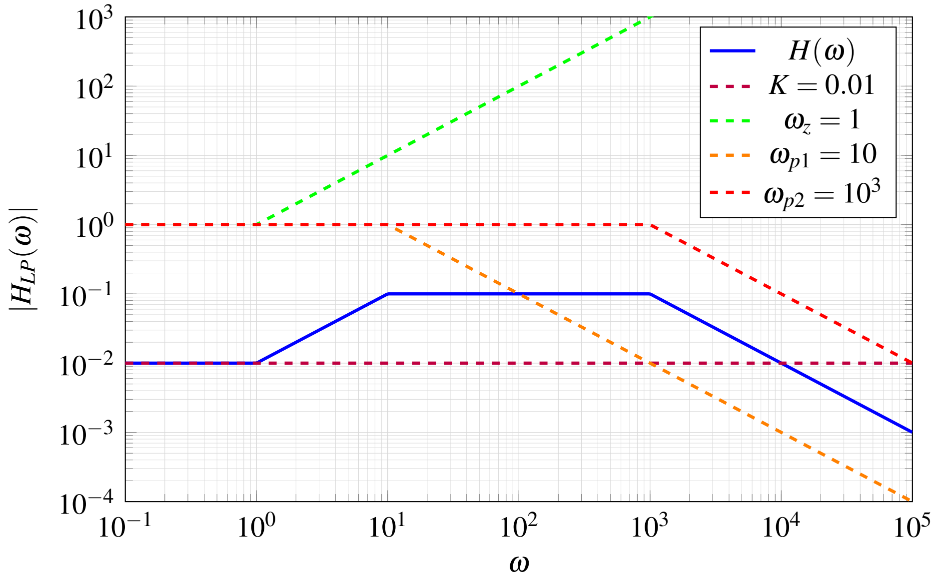
Now let's take a look at the Bode plot of a new transfer function.

$$H(\omega) = \frac{100(1 + j\omega)}{(j\omega)^2 + 1010(j\omega) + 10^4} \tag{11}$$

We must first factor it into its rational transfer function form:

$$H(\omega) = 0.01 \frac{(1 + j\omega)}{(1 + j\omega/10)(1 + j\omega/10^3)} \tag{12}$$

With the transfer function in its rational form, we see that $K = 0.01, \omega_z = 1, \omega_{p1} = 10, \omega_{p2} = 10^3$.

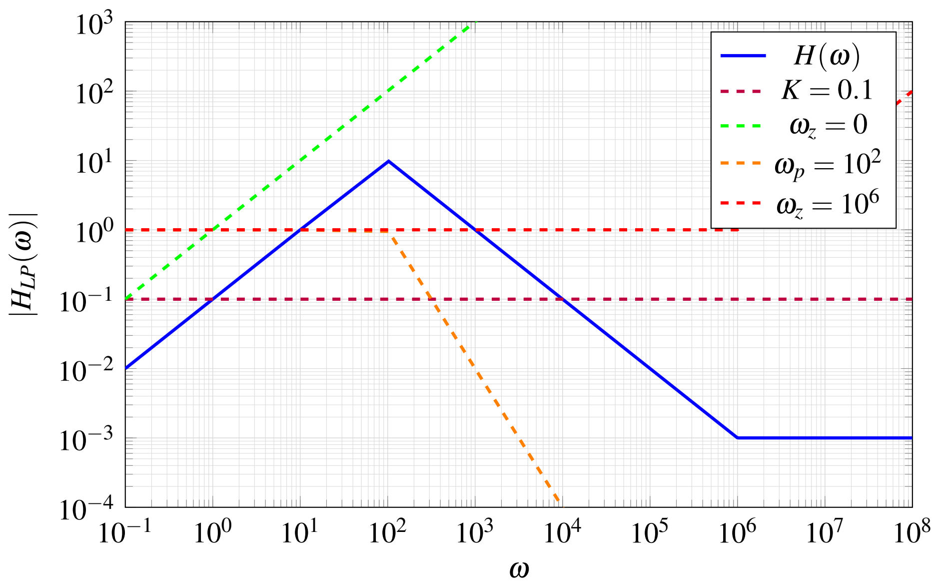


To provide an analysis for this Bode plot, we see that the plot starts off at $K = 0.01$. Then at $\omega_z = 1$, it starts rising with slope 1. When it hits the pole at $\omega_{p1} = 10$, the slope of 1 is cancelled out by the -1 slope that the pole provides. Then the Bode plot stays constant until $\omega_{p2} = 10^3$ at which it drops off with a slope of 1. We've provided Bode plots of the individual terms to give you a sense of how we "add" Bode plots together.

4.2 Zero at the Origin

In our final example, we examine the effects of a zero at the origin. Consider the following transfer function in rational form.

$$H(\omega) = 0.1 \frac{(j\omega)(1 + j\omega/10^6)}{(1 + j\omega/10^2)^2} \tag{13}$$



Since there is a zero at the origin, the plot initially starts with a slope of 1. There are no additional zeros or poles before $\omega = 1$, so we can approximate $\left|H(1)\right| = K = 0.1$. Then the double pole at $\omega_p = 10^2$ provides a slope of -2 cancelling with the current slope of 1 making the overall slope after ω_p equal to -1. Lastly, the zero at $\omega_z = 10^6$ provides an additional slope of 1 making $\left|H(\omega)\right|$ remains constant after ω_z.

5 Conclusion

All in all, Bode plots are a very powerful tool that let us approximate the behavior of a system without the use of numerical tools. While that on its own may seem unsatisfying, it was quite remarkable how accurate our approximations were.

In this note, we were developed a systematic approach to break down a transfer function as a product of its constituent components: zeros and poles. By doing so, we saw the effect of each individual zero and pole giving us a stronger understanding of how the two components interact and cancel out with each other. This understanding also let us develop and further understand higher-order filters such as the band-pass filter and n^{th} order low-pass filter.

In the next note, we will look into more filters involving inductors and a new phenomenon where the Bode approximation isn't quite as accurate. However, we will see how to take advantage of this behavior to design more powerful and interesting circuits.

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