EECS 16B Designing Information Devices and Systems II
Fall 2020 UC Berkeley **Note 9**

Introduction

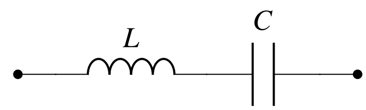
In the past couple of notes, we have developed techniques to analyze circuits in the frequency domain. This type of analysis let us design **filters** which satisfied a specific set of constraints and we were then able to plot **Bode plots** to better understand the behavior of our filters.

In this final note of the circuits module, we will be taking a look at filters composed out of RLC circuits. Recall that in the time domain, the pairing of the inductor and capacitor gave rise to oscillatory behavior. Upon analyzing our circuit in the frequency domain, we will notice a similar phenomenon called **resonance**.

While this may seem like an "undesirable" effect at first, we will come to appreciate it and learn how to take advantage of resonance to design stronger filters.

1 LC Tank

Let us start by looking at the impedance of an inductor and a capacitor in series.



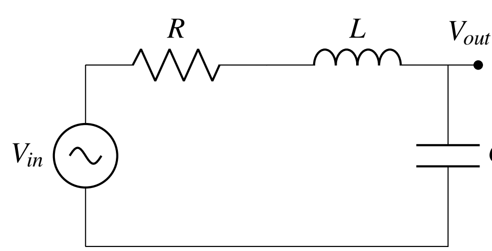
Since the two components are in series, we can compute Z_{total} by adding the two impedances

$$Z_{total} = j\omega L + \frac{1}{j\omega C} = \frac{(j\omega)^2 LC + 1}{j\omega C} \tag{1}$$

Notice that at the LC Tank's natural frequency $\omega_0 = \frac{1}{\sqrt{LC}}$, the impedance $Z_{total} = 0$. As a result, we define ω_0 as the **resonant frequency** of an LC tank as it will prefer to oscillate at this specific frequency.

2 RLC Filter

Let us analyze the transfer function $H(\omega) = \frac{\widetilde{V}_{out}}{\widetilde{V}_{in}}$ of a simple series RLC circuit



Using the voltage divider equation, the transfer function can be computed as follows

$$H(\omega) = \frac{Z_c}{Z_R + Z_L + Z_c} = \frac{1}{1 + j\omega RC + (j\omega)^2 LC} \tag{2}$$

Upon analyzing the transfer function for various values of ω, notice that this circuit behaves like a low-pass filter. However, before we dive into any further analysis, we will introduce the general form of a second-order system to develop the idea of damping.

3 Second Order Systems

Recall from the note on vector-differential equations, we defined the **damping ratio** ζ and **natural frequency** ω_n for any second-order differential equation.

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n \frac{dx}{dt} + \omega_n^2 x = u(t) \tag{3}$$

It turns out that in the frequency domain, we can define ζ and ω_n in a simlar fashion for any system with two poles:

$$H(\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} \tag{4}$$

This dual relation comes from taking the phasor transform of the differential equation given in (3) but we will omit its derivation here. In either case, comparing with the RLC transfer function, we see that

$$\omega_n = \frac{1}{\sqrt{LC}} \quad \zeta = \frac{1}{2}\frac{R}{\sqrt{L/C}} \tag{5}$$

This lets us defined damping in the frequency domain using the parameter ζ.

- For $0 < \zeta < 1$, we say the system is **underdamped.**
- For $\zeta = 1$, the system is **critically-damped.**
- For $\zeta > 1$, the system is **overdamped.**

At the moment, it may be unclear what damping in the frequency domain represents. We will get a better understanding of what this means by analyzing the Bode plots of $H(\omega)$ in the next section.

4 Damping in Frequency

Now that we have defined damping for a second-order system, let's take at the Bode plots of two RLC circuits with different values of R, L, and C. For illustrative purposes, we first compute $|H(\omega)|$ in terms of ω_n and ζ.

$$H(\omega) = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}$$

Now let us first look at a series RLC circuit that is underdamped meaning $\zeta < 1$.

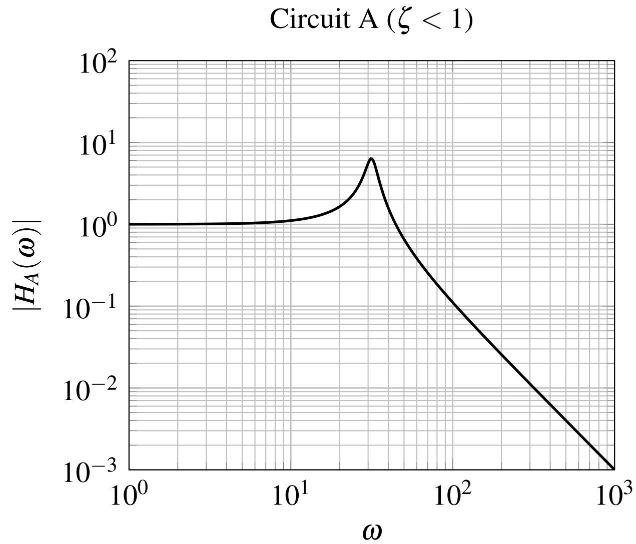


Figure 1: Underdamped RLC Bode Plot

Notice how there is a "spike" in the magnitude Bode plot. Finding the location of the spike is a nontrivial task that would involve taking the derivative of the denominator of $\left|H(\omega)\right|$ to minimize it. While we won't do this here, we claim that for values of $\zeta < \frac{1}{\sqrt{2}}$, the peak occurs at the frequency $\omega_n\sqrt{1-2\zeta^2}$.

For the purposes of this note, we will instead try to understand resonance and look at $H(\omega_n)$. By plugging in $\omega = \omega_n$, we see that $\left|H(\omega_n)\right| = \frac{1}{2\zeta}$. We will define this quantity $Q = \frac{1}{2\zeta}$ as the quality factor. Note that for $\zeta < \frac{1}{2}$, the resonant peak Q is greater than 1 meaning we can amplify signals at resonance!

Now let's take a look at the overdamped case meaning $\zeta > 1$.

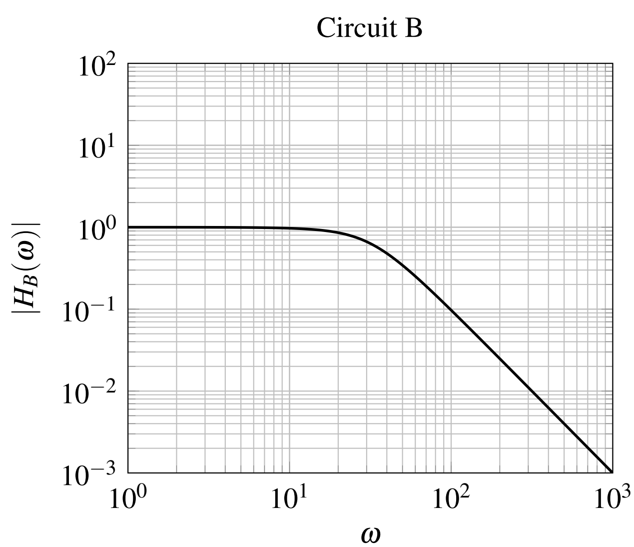


Figure 2: Overdamped RLC Bode Plot

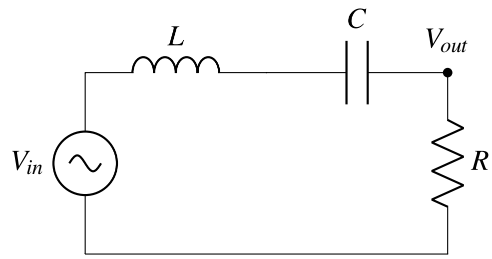
In this case, we see that the damping ratio is stronger than the natural frequency and we are unable to see a resonant peak. To verify this with our quality factor, notice that in the overdamped case, $Q < \frac{1}{2}$. Therefore, this tells us that for smaller values of Q we will not see a resonant peak.

5 Applications

Now that we have analyzed the resonant frequency and defined the quality factor of a circuit at resonance, we will take a look at a couple of applications.

5.1 Band-Pass Filter

By rearranging the order of our components in a series RLC circuit, we are able to build a band-pass filter. Consider the following RLC Circuit drawn below:

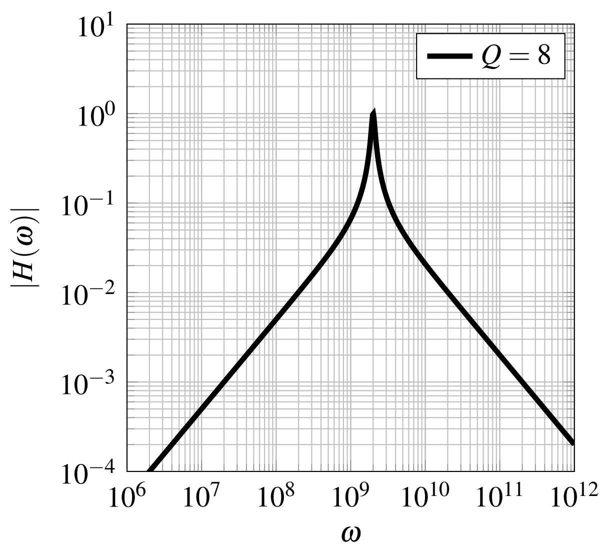


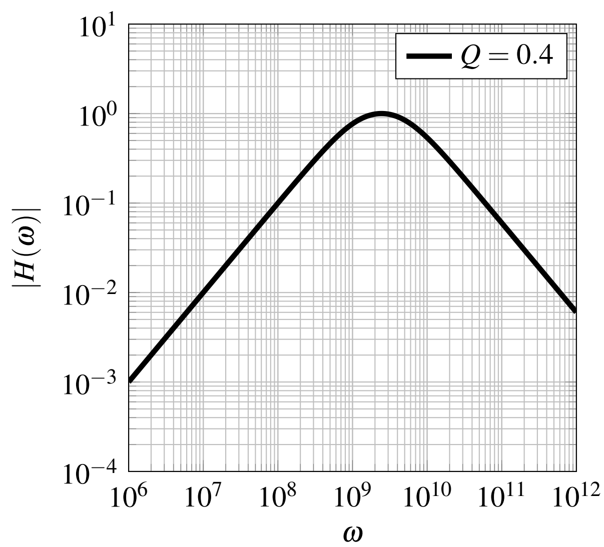
Its transfer function will be of the form

$$H(\omega) = \frac{j\omega RC}{1 + j\omega RC + (j\omega)^2 LC} = \frac{2\zeta\omega_n\omega}{\omega_n^2 + 2\zeta\omega_n\omega + (j\omega)^2} \tag{6}$$

Since the values for $\zeta = \frac{R}{2\sqrt{L/C}}$ and $\omega_n = \frac{1}{\sqrt{LC}}$ come from the denominator of a second-order system, they will remain the same for a series RLC circuit. However, the numerator of $H(\omega)$, $j\omega RC$, changes according to the values of ζ and ω_n.

There are multiple ways to analyze this circuit, but let us start by looking at a Bode plot of $\left|H(\omega)\right|$.



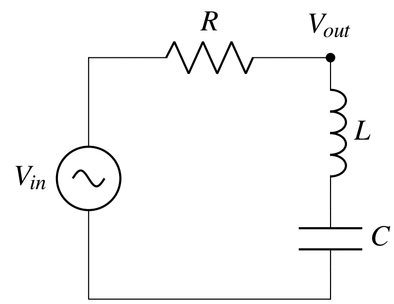


The plots above show the magnitude of $H(\omega)$ for the same $\omega_n = 2.44\,\text{GHz}$ but with different quality factors $Q = \frac{1}{2\zeta}$. This type of circuit is very desirable if we would like to capture a very small range of frequencies.

Notice when Q is large we get a very sharp peak at the resonance frequency and magnitude drops off quickly rejecting all other frequencies. On the other hand, for smaller values of Q, we are unable to see a resonant peak and the magnitude decays much slower.

5.2 Notch Filter

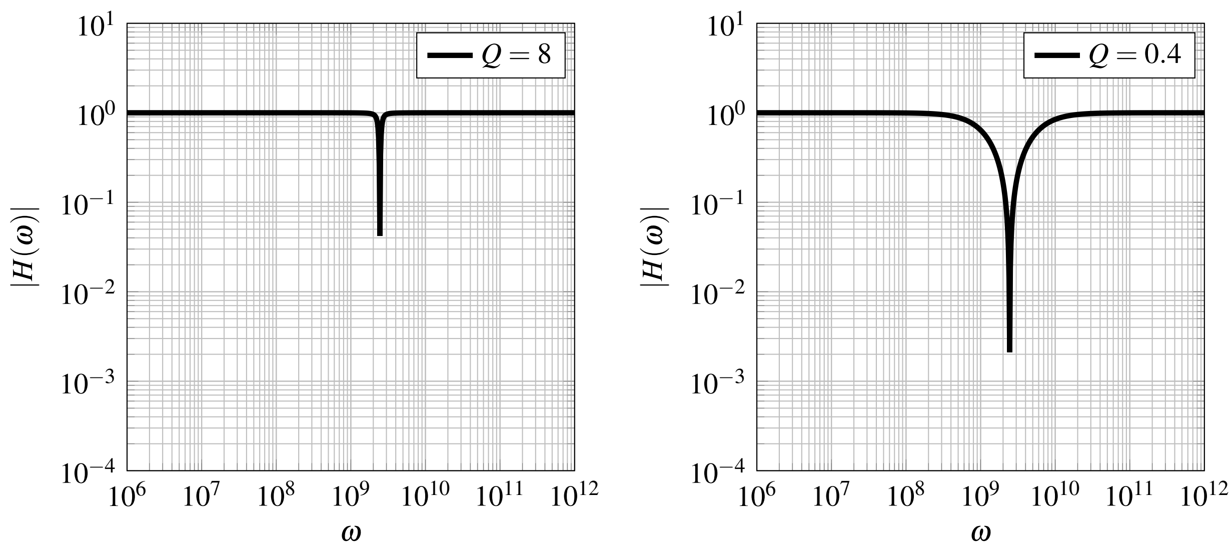
Using a very similar idea to the band-pass filter we just designed, we are also able to design a **Notch Filter** by taking V_{out} to be the voltage across the series combination of the inductor and capacitor.



The transfer function will be of the form

$$H(\omega) = \frac{1+(j\omega)^2 LC}{1+j\omega RC+(j\omega)^2 LC} = \frac{\omega_n^2+(j\omega)^2}{\omega_n^2+2\zeta\omega_n\omega+(j\omega)^2} \tag{7}$$

Let's take a look at the magnitude Bode plots to see how our filter behaves.



This filter lets all frequenices through except for the resonant frequency $\omega_n = \frac{1}{\sqrt{LC}}$. Do note that $H(\omega) = 0$ at the resonant frequency but due to numerical approximations, the plots are unable to show the dropoff to $-\infty$ on a log scale. In addition, we can see that for smaller values of Q, the filter drops off much slower.

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