EECS 16B Fall 2020 — Designing Information Devices and Systems II, UC Berkeley — Note j

1 Introduction

Most students have some basic background in complex numbers ($\mathbb{C}$) from high school. Traditionally, complex numbers are introduced in order to motivate the existence of solutions to polynomial equations. We'll however, be taking a different approach and in following notes see how complex numbers play a crucial role in modeling real world phenomena such as rotations, oscillations, and periodicity.

The purpose of this note is to solidify our understanding of complex numbers. Let's begin with the most basic definition: $j = \sqrt{-1}$. In most engineering discplines, we will use j, so as not confuse ourselves with current or the identity matrix I.

2 Complex Numbers

Any complex number $z \in \mathbb{C}$ can be represented in two form $z = a + bj$, where a is the **real part** and b is the **imaginary part**. This form is referred to as **rectangular form**, and as we will see, addition in this form is very easy and akin to vector addition.

The **complex conjugate** of z, represented by $\bar{z}$ (and sometimes by z^*), is defined as follows:

$$\bar{z} = \overline{(a+bj)} = a - bj \tag{1}$$

The **magnitude** of a complex number, z, is given by

$$|z| = \sqrt{a^2 + b^2} \tag{2}$$

and the angle or **phase** is given by

$$\theta = \angle z = \text{atan2}(b, a) \tag{3}$$

Here, atan2(y, x) is a function[1] that returns the angle from the positive x-axis to the vector from the origin to the point (x, y).

2.1 Complex Plane

The complex plane lets us visualize complex numbers as vectors in a two dimensional space by adding another independent axis to represent the purely imaginary numbers. As shown in Figure 1, a complex number $z = x + jy$ has an intercept at x along the **real axis** and y along the **imaginary axis**. We can also see visually that the magnitude of z is its distance from the origin, and the phase is the angle from the positive real axis.

[1] See its relation to $\tan^{-1}\left(\frac{y}{x}\right)$ at `https://en.wikipedia.org/wiki/Atan2`.

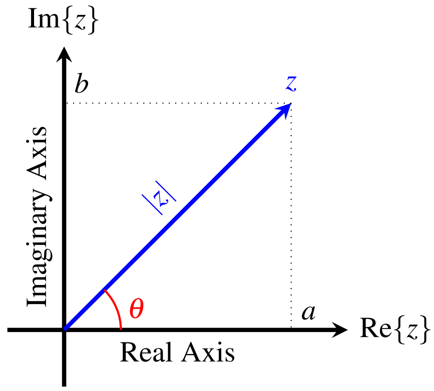


Figure 1: Complex number $z = a + bj$ depicted as a vector in the complex plane.

2.1.1 Warning in Advance

However, we must be careful with this visualization. While there is a 1-1 mapping from $\mathbb{C}$ to $\mathbb{R}^2$, the two structures behave different *algebraically*. We'll examine this behavior in the next part by looking at how complex numbers add, multiply, and divide.

2.2 Basic Operations

Let's say we have two complex numbers $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$.

2.2.1 Addition

You may remember that the addition of complex numbers is defined as follows:

$$z_1 + z_2 = (a_1 + a_2) + j(b_1 + b_2) \tag{4}$$

The rules of addition and subtraction behave identically to vectors in $\mathbb{R}^2$. The complex numbers 1 and j act as the standard basis vectors $\vec{e}_1, \vec{e}_2$ of $\mathbb{R}^2$.

2.2.2 Multiplication

Multiplication is a little more complicated. It behaves very similar to polynomial multiplication, except the indeterminate (i.e. the variable) is replaced by j, and we have j^2 = -1:

$$\begin{aligned} z_1 \times z_2 &= (a_1 + jb_1) \times (a_2 + jb_2) && (5) \\ &= a_1 * a_2 + jb_1 * a_2 + ja_1 * b_2 + j^2 b_1 * b_2 && (6) \\ &= (a_1 * a_2 - b_1 * b_2) + j(a_1 * b_2 + a_2 * b_1) && (7) \end{aligned}$$

This is where complex numbers begin to behave different from vectors in $\mathbb{R}^2$. Notice how this is much different from say element-wise multiplication of the two vectors:

$$z_1 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \quad z_2 = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$$

2.2.3 Division

Before we move on to division, let's see what the multiplicative inverse would look like:

$$\begin{aligned} \frac{1}{z} &= \frac{1}{a+jb} = \frac{1}{a+jb} \times \frac{a-jb}{a-jb} \\ &= \frac{a-jb}{a^2+b^2} = \frac{a}{a^2+b^2} - j\frac{b}{a^2+b^2} \end{aligned} \tag{8}$$

Above, we multiply both the numerator and denominator by $\bar{z}$. This allows us to make the denominator real, and we also observe $\bar{z} \times z = |z|^2$. Following the same train of thought, let's define division as well:

$$\frac{z_1}{z_2} = \frac{a_1+jb_1}{a_2+jb_2} \tag{9}$$

$$= \frac{(a_1+jb_1)\times(a_2-jb_2)}{a_2^2+b_2^2} \tag{10}$$

$$= \frac{(a_1 * a_2 + b_1 * b_2) - j(a_1 * b_2 - a_2 * b_1)}{a_2^2+b_2^2} \tag{11}$$

At line (10), we subsititute the multiplicative inverse found in (8), and we continue by carrying out the multiplication as defined in (7).

3 The Polar Form

The polar form of z is a very important representation as it simplifies multiplication. The polar form of $z = a + jb$ is given as follows:

$$z = re^{j\theta}$$

where r represents the **magnitude** $|z|$ and θ represents the **phase**. This is an alternate way to represent the same complex number z and the figure below compares polar and rectangular form side by side:

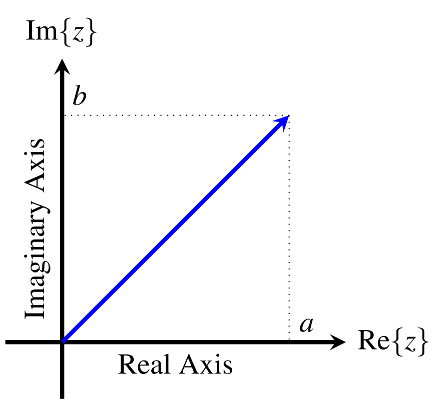


Figure 2: Rectangular Form: $z = a + bj$

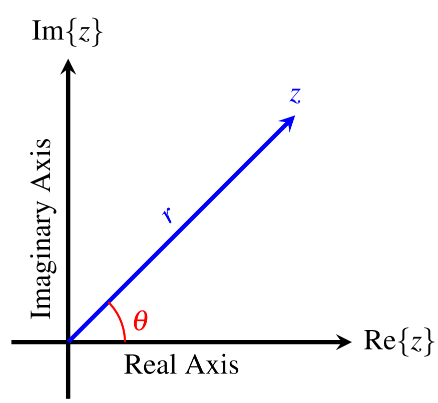


Figure 3: Polar Form: $z = re^{j\theta}$

Multiplication in this form is given as:

$$z_1 \times z_2 = (r_1 e^{j\theta_1}) \times (r_2 e^{j\theta_2}) = (r_1 \cdot r_2) e^{j(\theta_1+\theta_2)}$$

However, you may notice that addition is quite difficult in polar form. In the next part, we will look at how to convert between polar and rectangular form.

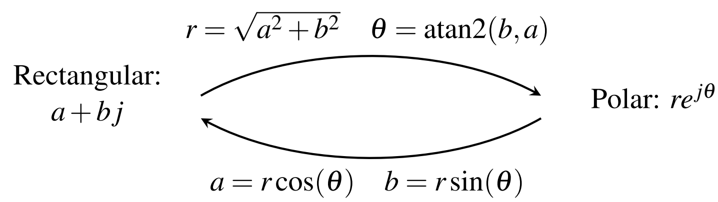
3.1 Euler's Formula

One of the most important relationships in our study of complex numbers is Euler's formula which says

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \tag{12}$$

We can extrapolate this formula by looking at Figure 3 and applying trigonometric relations.

Euler's Formula lets us convert a complex number represented in polar form into rectangular form. The figure below shows how to convert a complex number from polar to rectangular form and vice-versa.



Since addition is easier in rectangular form whereas multiplication is easier in polar form, we will often switch between forms to make arithmetic operations more convenient.

3.2 Revisiting Multiplication

In thi section, we develop more intuition on how we can realize that complex multiplication is a rotation operation, followed by scaling.

Let's see how a complex number $z = 1 + j0$ changes as we multiply it with $z_1 = 1 + j = \sqrt{2}e^{j45^\circ}$ repeatedly:

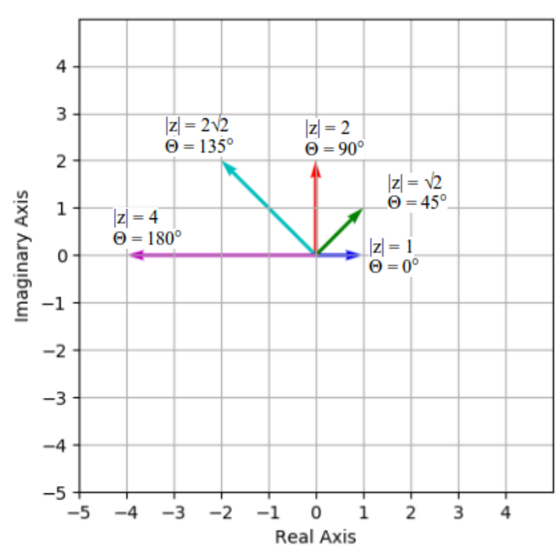


Figure 4: Multiplying z (in blue) by z_1 repeatedly

We can view multiplication by z_1 as a rotation by $\angle z_1 = 45^\circ$ followed by a scaling of $|z_1| = \sqrt{2}$. This should show the power of polar form when it comes to multiplication: We can view any complex number as a rotation and scaling operation.

4 Developing Euler's Formula

In the last sections we stated,

$$z = re^{j\theta}. \tag{13}$$

But what does $e^{j\theta}$ even mean? It is a natural first step, from our experiences, to use the exponential's definition in the form of its Taylor's expansion around 0 (or the Maclaurin's expansion of '$e^{\cdot}$'):

$$\begin{aligned} z &= re^{j\theta} && (14)\\ &= r\left(1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \cdots + \frac{(j\theta)^{2n}}{2n!} + \frac{(j\theta)^{2n+1}}{(2n+1)!} + \cdots\right) && (15)\\ &= r\left(1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \cdots + (-1)^n\frac{\theta^{2n}}{2n!} + j(-1)^n\frac{\theta^{2n+1}}{(2n+1)!} + \cdots\right) && (16)\\ &= r\left[\left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4} - \cdots + (-1)^n\frac{\theta^{2n}}{2n!} + \cdots\right) + j\left(\theta - \frac{\theta^3}{3} + \cdots + (-1)^n\frac{\theta^{2n+1}}{(2n+1)!} + \cdots\right)\right] && (17)\\ &= r\left(\cos(\theta) + j\sin(\theta)\right). && (18) \end{aligned}$$

Here, $\theta = \angle z$ and the angles are clearly measured in radians. At (16), we used the fact that j^k has a cyclic pattern to it: $j^0 = +1$, $j^1 = +j$, $j^2 = -1$, $j^3 = -j$, $j^4 = +1$, and so it goes. Equating the first and last lines, we get Euler's Formula from the previous section.

Concept Check: Using Euler's equation:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \tag{19}$$

write sine and cosine as sums of complex exponentials.
Solution:

$$\begin{aligned} &e^{-j\theta} = \cos(-\theta) + j\sin(-\theta) = \cos(\theta) - j\sin(\theta) && (20)\\ &\text{Adding and Subtracting equation (19) and (20), we get :} \\ &2\cos(\theta) = e^{j\theta} + e^{-j\theta} \Rightarrow \cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} && (21)\\ &2j\sin(\theta) = e^{j\theta} - e^{-j\theta} \Rightarrow \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} = \frac{-je^{j\theta} + je^{-j\theta}}{2} && (22) \end{aligned}$$

This is perhaps the most important fact about complex numbers that we'll be using in the following notes. Notice how we are able to sinusoidal function and decompose it into a sum of complex exponentials, both of which represent complex numbers. This should give more insight on how complex numbers are able to model oscillatory behavior.

We end our discussion of Euler's Formula by bringing up the famous identity famous identity $e^{j\pi} + 1 = 0$, connecting five very fundamental numbers together: 0 (the additive identity), 1 (the multiplicative identity), e (the base of the natural logarithm, defined because we want a function whose derivative was itself), j (the basic imaginary number $\sqrt{-1}$), and π (the area of a perfect circle with radius 1).

5 Useful Identities

Complex Number Properties

Rectangular vs. polar forms: $z = x + jy = |z|e^{j\theta}$

where $|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$, $\theta = \text{atan2}(y, x)$. We can also write $x = |z|\cos\theta$, $y = |z|\sin\theta$.

Euler's identity: $e^{j\theta} = \cos\theta + j\sin\theta$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}, \ \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

Complex conjugate: $\overline{z} = x - jy = |z|e^{-j\theta}$

$$\overline{(z+w)} = \overline{z} + \overline{w}, \ \overline{(z-w)} = \overline{z} - \overline{w}$$

$$\overline{(zw)} = \overline{z}\overline{w}, \ \overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$$

$$\overline{z} = z \Leftrightarrow z \text{ is real}$$

$$\overline{z} = -z \Leftrightarrow z \text{ is purely complex, i.e. no real part}$$

$$\overline{(z^n)} = (\overline{z})^n$$

Complex Algebra

Let $z_1 = x_1 + jy_1 = |z_1|e^{j\theta_1}$, $z_2 = x_2 + jy_2 = |z_2|e^{j\theta_2}$.

(Note: we adopt the easier representation between rectangular form and polar form.)

Addition: $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$

Multiplication: $z_1 z_2 = |z_1||z_2|e^{j(\theta_1+\theta_2)}$

Division: $\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}e^{j(\theta_1-\theta_2)}$

Power: $z_1^n = |z_1|^n e^{jn\theta_1}$

$z_1^{\frac{1}{2}} = \pm|z_1|^{\frac{1}{2}} e^{j\frac{\theta_1}{2}}$

(Note: Just like square roots are not unique, other fractional powers of z_1 are not unique as well)

Useful Relations

$$-1 = j^2 = e^{j\pi} = e^{-j\pi}$$

$$j = e^{j\frac{\pi}{2}} = \sqrt{-1}$$

$$-j = -e^{j\frac{\pi}{2}} = e^{-j\frac{\pi}{2}}$$

$$\sqrt{j} = (e^{j\frac{\pi}{2}})^{\frac{1}{2}} = \pm e^{j\frac{\pi}{4}} = \frac{\pm(1+j)}{\sqrt{2}}$$

Concept Check: Verify the above identities for yourself if you have not done so in prior classes.

6 Conclusion

The polar form of z is a very important representation as it greatly simplifies multiplication. The polar form of $z = a + jb$ is given as follows:

$$z = re^{j\angle\theta}$$

We can also represent the conjugate of z is given as $\overline{z} = re^{-j\theta}$ since $\sin(-\theta) = -\sin(\theta)$ while $\cos(-\theta) = \cos(\theta)$ from trigonometry.

When multiplying two complex numbers z_1 and z_2 in polar form, we multiply their magnitudes and add their phase.

$$z_1 \times z_2 = (r_1 e^{j\theta_1}) \times (r_2 e^{j\theta_2}) = (r_1 \cdot r_2)e^{j(\theta_1+\theta_2)}$$

In Section 3.2 we developed the intuition that multiplication by a complex number is a rotation operation followed by scaling. Note that we are defining rotation by a positive number in a counter-clockwise direction. We will further explore this idea in the next bonus section when we create a matrix representation of complex numbers and connect it to the rotation matrix from 16A.

7 (Bonus) Complex Numbers modeled using Matrices

Viewing complex numbers as vectors definitely seems attractive and it does fit into our visualization of the complex plane, but it has a major flaw — vectors do not naturally multiply, but complex numbers do. In fact, multiplication is the *raison d'etre* for complex numbers. So, how do we get a better model? What both adds and multiplies? Enter matrices, and more specifically scaled rotation matrices.

7.1 Matrix form of rotations

But first, what is a rotation matrix? To begin answering this question, we need to first understand what a rotation tranformation would look like. Rotating the vector $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ by angle θ in the counter clockwise direction would give us $\widetilde{\vec{e}_1} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$, and similarly for $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we will get $\widetilde{\vec{e}_2} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$. Hence, we can describe the rotation transform (by angle θ) as the following matrix:

$$\widetilde{\vec{v}} = R_\theta \vec{v} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \vec{v} \tag{23}$$

An important thing to note is this rotation matrix has orthonormal columns[2]. Next, what would happen if we rotated a vector by θ_1 and then by θ_2? Well, it would be equivalent to rotating it by $\theta_1 + \theta_2$, hence we have:

$$R_{\theta_1} * R_{\theta_2} = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{bmatrix} = \begin{bmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix} = R_{\theta_1 + \theta_2} \tag{24}$$

Concept Check: Use basic trigonometry (in particular, the sum-angle formulas for sine and cosine that you probably derived in high school) to check the equality established in equation (24). Futhermore, rotations in 2D are commutative[3] as well. Show that this is true by proving $R_{\theta_1} * R_{\theta_2} = R_{\theta_2} * R_{\theta_1}$

When we look back at rotation matrix in (23), it bears some resemblance to the Euler form (equation 18) we discovered in the previous section. If we have a complex number $z = a + jb = \cos(\theta) + j\sin(\theta)$, where $|z| = 1$ (for simplicity, we will look at scaling a bit later) and $\angle z = \theta$, then we could define a matrix $Z_{(a,b)}$ as follows:

$$Z_{(a,b)} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \tag{25}$$

Concept Check: Check that this matrix has orthogonal columns.

We can express the fundamentally two-dimensonal nature of such matrices by expressing them using a clear basis:

$$Z_{(a,b)} = aI + bJ \text{ where } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{26}$$

[2]The columns in a matrix with orthonormal columns all have norm 1 and are mutually orthogonal to each other (i.e. their inner products with each other are zero). Such matrices are commonly referred to as **orthogonal** matrices in the mathematical literature.

[3]This commutative property for rotations only holds for 2D spaces, and not for 3D spaces. Take a second to think about this!

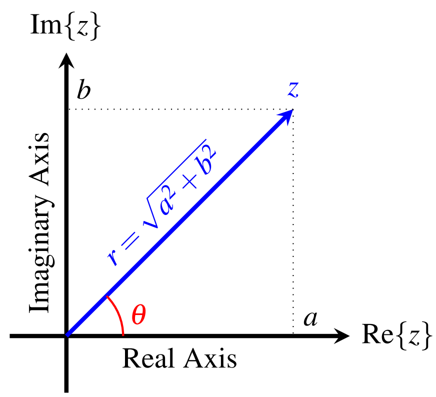
Notice that $J^2 = -I$ above, and so the matrix J acts like the counterpart of the basic imaginary number j.

What is a complex conjugate in this representation? What can we do to swap the b and $-b$ in the matrix above? Indeed we see that **transposing** the matrix corresponds to complex conjugation of the underlying complex number. It has no effect on a scaled identity matrix which would correspond to a purely real number. But $J^T = -J$.

Next, let's look at the scaling. In this case, we have $z = a + jb$, with $|z| = \sqrt{a^2+b^2}$. To account for this in our matrix model, we can factor out $|z|$ as follows:

$$Z_{a,b} = \sqrt{a^2+b^2}\begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}} & -\frac{b}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{bmatrix} \tag{27}$$

Looking at the above form, the factor out in the front is responsible for the scaling. From the figure below, we can find $\theta = \text{atan2}(b,a)$ such that $\cos(\theta) = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin(\theta) = \frac{b}{\sqrt{a^2+b^2}}$.



Now, let's see if this model fits with everything that we know about complex arithmetic.

7.1.1 Addition:

For two complex numbers, $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$, we have:

$$Z_{(a_1,b_1)} + Z_{(a_2,b_2)} = \begin{bmatrix} a_1+a_2 & -(b_1+b_2) \\ b_1+b_2 & a_1+a_2 \end{bmatrix} = Z_{(a_1+a_2,b_1+b_2)}$$

Hence, it satisfies our definition of addition.

7.1.2 Multiplication by real number:

Let $z = a + jb$, then $\lambda z = \lambda a + j\lambda b$, where λ is a real number. This can be easily extended to our matrix form as well:

$$\lambda Z_{(a,b)} = \lambda \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} \lambda a & -\lambda b \\ \lambda b & \lambda a \end{bmatrix} = Z_{(\lambda a,\lambda b)}$$

7.1.3 Multiplication by a complex number:

Finally, and the reason we are pursuing this representation, multiplication by another complex number. Let $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$, then we have $z_1 \times z_2 = (a_1 * a_2 - b_1 * b_2) + j(a_1 * b_2 + a_2 * b_1)$. Let's check if this is the case with matrix multiplication:

$$Z_{(a_1,b_1)} * Z_{(a_2,b_2)} = \begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix} \begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix} = \begin{bmatrix} a_1 * a_2 - b_1 * b_2 & -(a_1 * b_2 + a_2 * b_1) \\ a_1 * b_2 + a_2 * b_1 & a_1 * a_2 - b_1 * b_2 \end{bmatrix} = Z_{(a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1)}$$

Note that since our 2D rotations are communtative, so are the multiplications with complex numbers.

Multiplication by the complex conjugate also clearly gives an identity matrix with the magnitude squared along the diagonal.

It turns out, although we will not show this here, that even the natural generalization of exponentiation to matrices works with this matrix model for complex numbers. We get $e^{a+jb} = e^a e^{jb} = e^a(\cos b + j \sin b)$. Actually, the understanding of the natural generalization of exponentiation to matrices requires understanding the solutions to systems of differential equations, where the complex exponentiation case turns out to represent the behavior of RLC circuits. To understand this requires understanding the eigenvalues of the kinds of matrices we find here, but that is a subject of a different note.

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