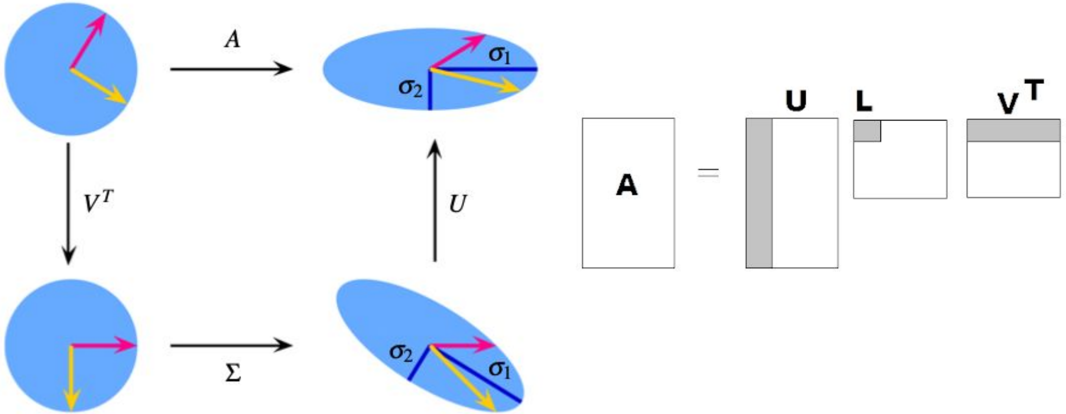
Logo credits go to Moses Won

Discussion 10B

Geometry of the SVD



Recap

The **Singular Value Decomposition** of an mxn matrix A with rank r is:

$\sigma_i = 0$
for $i > r$

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \ldots + \sigma_r \vec{u}_r \vec{v}_r^T = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$$

$$+ \sigma_{r+1} \vec{u}_{r+1} \vec{v}_{r+1}^T + \ldots + \sigma_n \vec{u}_n \vec{v}_n^T$$

Each term $\boldsymbol{\sigma_i u_i v_i^T}$ is a **rank 1 matrix.**

- Vectors $\mathbf{u_i}$ are orthonormal and called **left-singular vectors.**
- Scalars $\boldsymbol{\sigma_i}$ are non-negative and called **singular values.**
- Vectors $\mathbf{v_i}$ are orthonormal and called **right-singular vectors.**

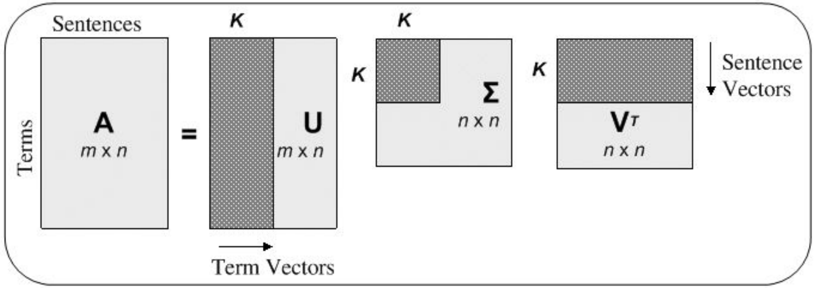
Full SVD

We can also write out the SVD as a product of matrices:

$$A = U_1 S V_1^T$$

Compact SVD

$S = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}$



Full SVD

U = [$\mathbf{U_1}$ $\mathbf{U_2}$] is an **m x m** square matrix with orthonormal columns.

$\mathbf{\Sigma} = \begin{bmatrix} S_{r\times r} & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix}$ is an **m x n** matrix and **S** is an **r x r** diagonal matrix.

V: [$\mathbf{V_1}$ $\mathbf{V_2}$] is an **n x n** square matrix with orthonormal columns.

Computing the Full SVD

but if $m < n$
then AA^T will be more computationally efficient, (λ_i, u_i)

$v_i = \frac{A^T u_i}{\sigma_i}$

We can compute the SVD through the following procedure:

1. Compute eigenvalues and eigenvectors ($\boldsymbol{\lambda}_i$, $\mathbf{v}_i$) of A^TA.
2. The singular values $\boldsymbol{\sigma}_i = \sqrt{\lambda_i}$
3. Compute left-singular vectors $\mathbf{u_i} = \frac{\mathbf{Av_i}}{\sigma_\mathbf{i}}$

   Use $Av_i = \sigma_i u_i$

   for $i = 1, \dots, r$
4. For $i > r$, $\boldsymbol{\sigma}_i = \mathbf{0}$ and vectors $\mathbf{u}_i$ and $\mathbf{v}_i$ span the **Nul A** and **Nul A^T**.
   a. In other words, we can compute $\mathbf{u}_i$ by finding an orthonormal basis for the null-space of $\mathbf{A}^T$ and $\mathbf{v}_i$ from the null-space of $\mathbf{A}$

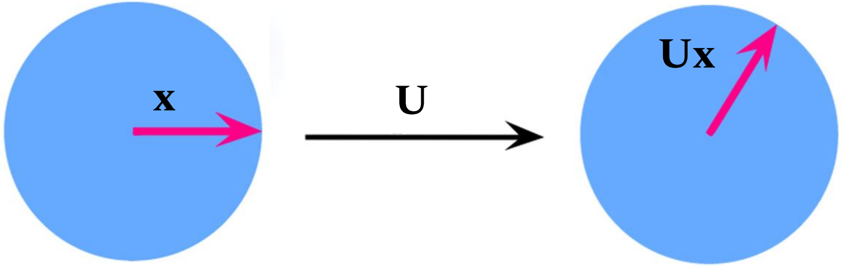
$\{u_{r+1}, \dots, u_n,\}$ we can compute an orthonormal basis for Nul A^T

$\{v_{r+1}, \dots, v_r\}$ " " for Nul A

Full SVD

"unitary"

The $\boxed{\mathbf{U}}$ and $\boxed{\mathbf{V}}$ matrices are square and have orthonormal columns.



$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

An orthonormal transform **rotates** a vector **x.** A rotation does change the norm of a vector.

$$\|x\| = \|Ux\|$$

Full SVD

Since $\mathbf{A} = \mathbf{U\Sigma V^T}$ we can view $\mathbf{A}$ as three operations:

A is a linear transformation that maps vectors $\mathbf{x}$ to $\mathbf{y} = \mathbf{Ax} = \mathbf{U\Sigma V^T x}$

$A = U \Sigma V^T$

1. $\mathbf{V^T}$ rotates the vector $\mathbf{x}$
2. $\mathbf{\Sigma}$ scales the vector $\mathbf{V^T x}$
3. $\mathbf{U}$ rotates $\mathbf{\Sigma V^T x}$