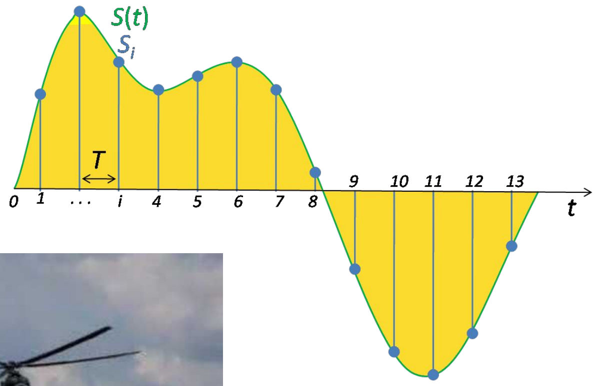
Discussion 12B

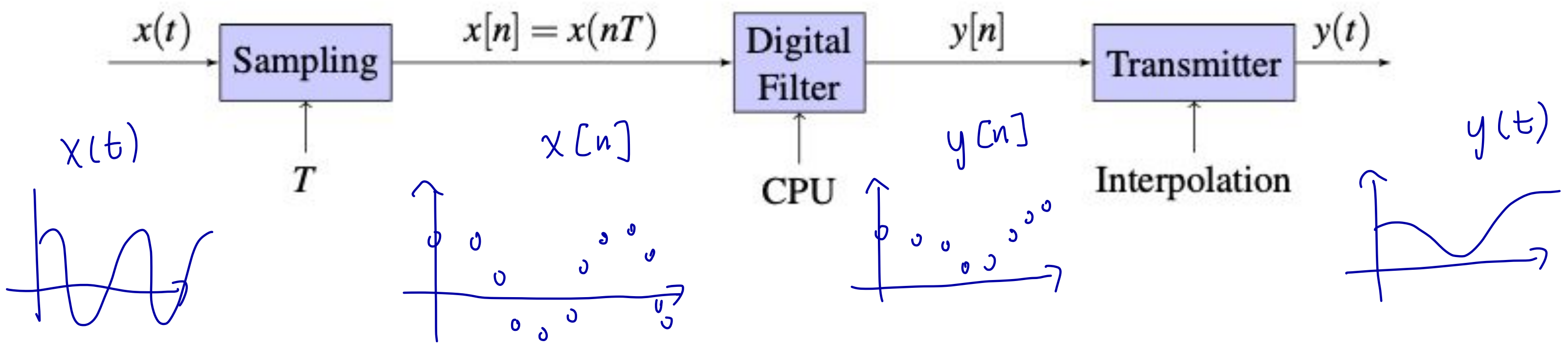
Sampling & Aliasing

Logo credits go to Moses Won



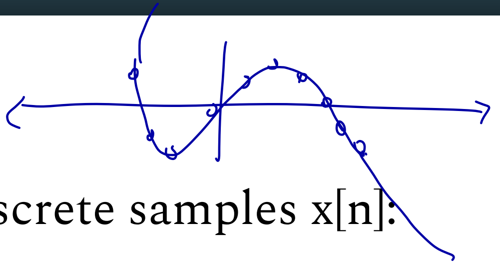
Recap

Final Module on Digital Signal Processing!

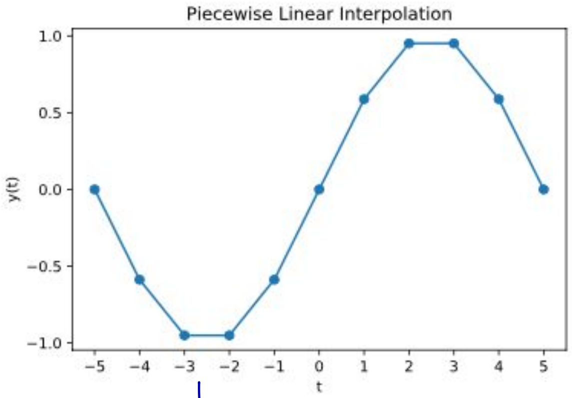


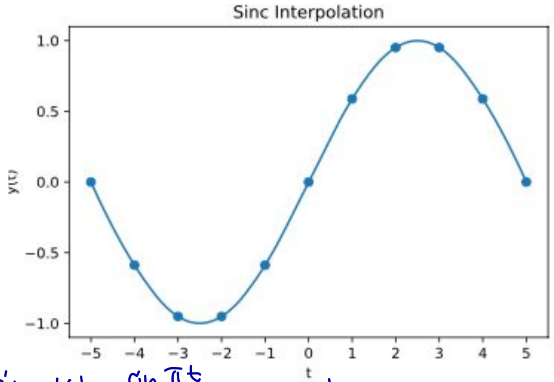
- The real world behaves in continuous-time but computers behave in discrete-time.
- On Monday's discussion we saw how to reconstruct a discrete signal into a continuous function using **interpolation**.

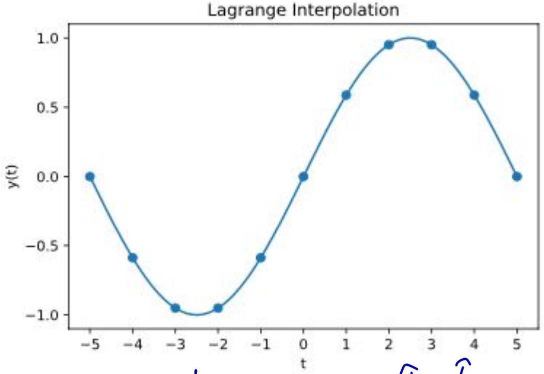
Interpolation

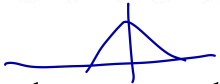


We introduced multiple ways to interpolate a set of discrete samples x[n]:

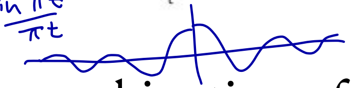


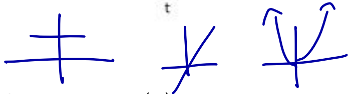






$\text{sinc}(t) = \frac{\sin \pi t}{\pi t}$





This was done by taking a linear combination of **basis functions** ϕ(t)

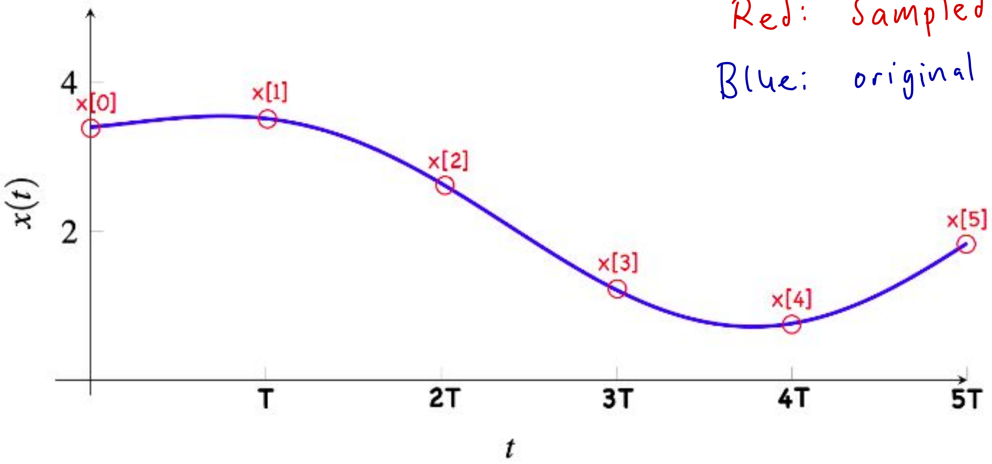
$$y(t) = \sum_{k=0}^{N-1} y[k]\phi(t - kT),$$

Sampling

Today, we're going to focus on **sampling** and a phenomenon called **aliasing.**

Red: Sampled

Blue: original CT signal



Given a continuous signal x(t), we take "samples" by evaluating it every $\mathbf{T_s}$ secs.

$\mathbf{T_s}$ is the sampling **period** and $\boldsymbol{\omega_s} = \mathbf{2\pi / T_s}$ is the sampling **frequency.**

Shannon-Nyquist Theorem

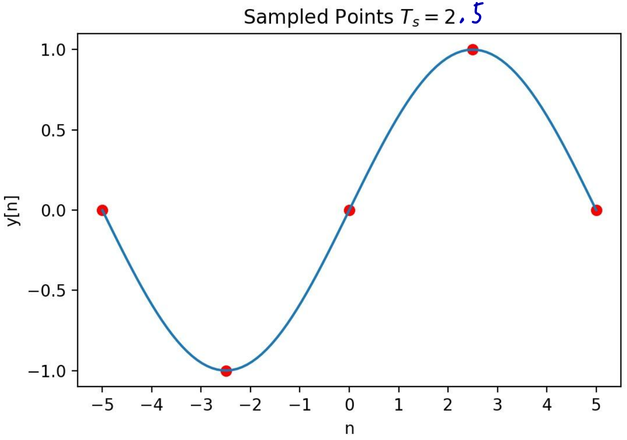
$x(t) = \sin(4\pi t) \quad \omega = 4\pi$

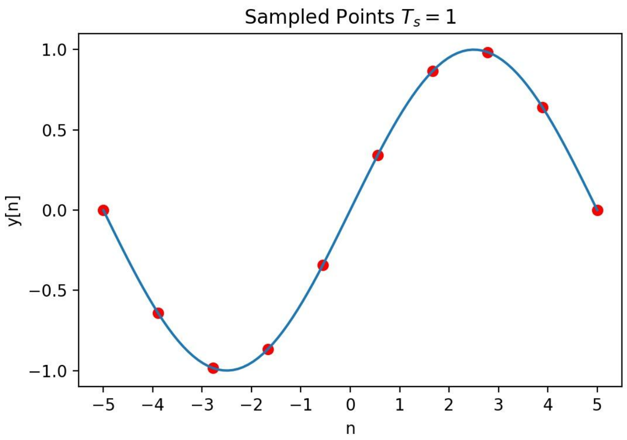
$y(t) = \cos(6\pi t) + \sin(4\pi t) \quad \omega_{max} = 6\pi$

Given a CT signal x(t) with maximum frequency $\boldsymbol{\omega}_{\mathbf{max}}$, we can always reconstruct the original signal through **sinc interpolation** if we sample at frequency $\boldsymbol{\omega}_{\mathbf{s}} > \mathbf{2}\boldsymbol{\omega}_{\mathbf{max}}$.

$\omega_s = \frac{2\pi}{T_s} > 2\omega_{max}, \quad T_s < \frac{\pi}{\omega_{max}}$

to guarantee a perfect reconstruction



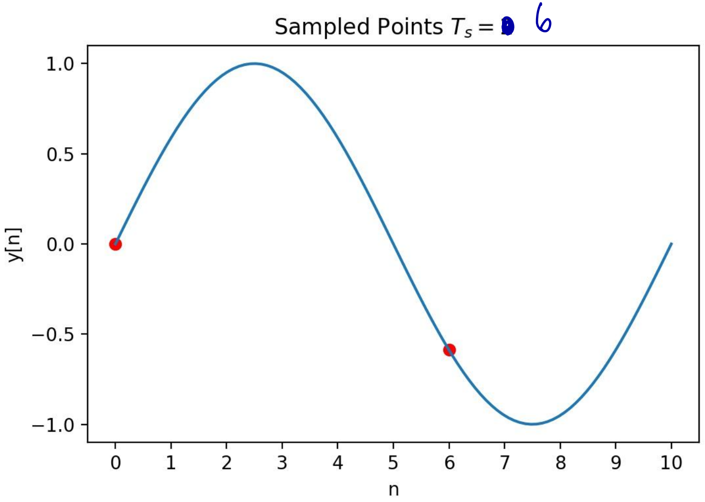


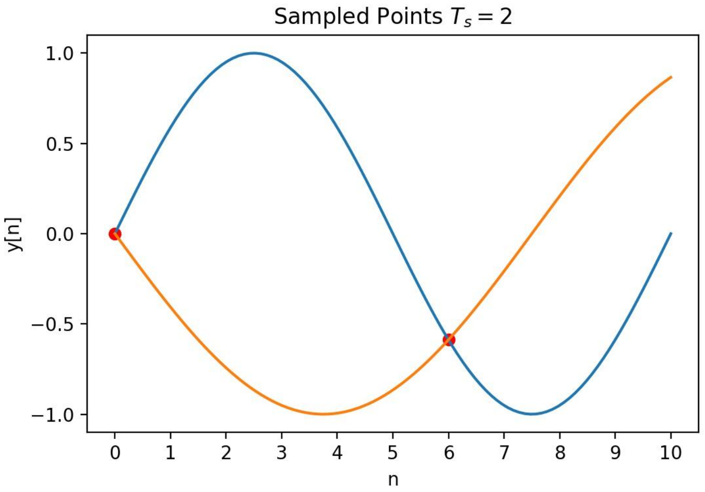
What happens when we don't sample fast enough?

Aliasing

If we sample with frequency $\boldsymbol{\omega_s < 2\omega_{max}}$, we will see "aliasing" when reconstructing the signal with sinc interpolation.

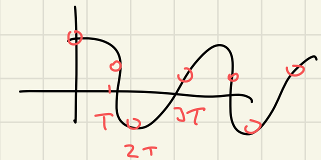
Blue: original
Orange: Reconstructed





Brute Force Sinc Interpolation

Given a sinusoid $x(t) = \cos(\underset{\text{freq}}{\underset{\uparrow}{w_0}} t + \underset{\text{phase}}{\underset{\uparrow}{\phi}})$ sampled at rate T



1. The discrete samples will $x[n] = \cos(w_0 nT + \phi)$ substitution $t = nT$

2. Sinc Interpolation picks the reconstruction with lowest frequency

- The "simple" option is $x(t) = \cos(w_0 t + \phi)$ by substituting $n = \frac{t}{T}$
- Notice that $x[n] = \cos(2\pi n - w_0 nT - \phi)$ because $\cos(2\pi n - x) = \cos(x)$

$$= \cos((2\pi - w_0 T)n - \phi)$$

Another option is $x(t) = \cos\left((2\pi - w_0 T)\frac{t}{T} - \phi\right)$

$$= \cos\left(\left(\frac{2\pi}{T} - w_0\right)t - \phi\right)$$

If $T >= \frac{\pi}{w_0}$, $w_0 >= \frac{\pi}{T}$, $2w_0 \geq \frac{2\pi}{T}$ so we can say that $w_0 \geq \frac{2\pi}{T} - w_0$

we get an "aliased" signal $\cos\left(\left(\frac{2\pi}{T} - w_0\right)t - \phi\right)$

If $T < \frac{\pi}{w_0}$, then $w_0 < \frac{\pi}{T}$, so $2w_0 < \frac{2\pi}{T}$ we can say $w_0 < \frac{2\pi}{T} - w_0$

$x(t) = \cos\left(\frac{\pi}{5} t\right)$, $\quad \omega_{max} = \frac{\pi}{5}$ so Sampling Thm says $T_s < \frac{\pi}{\pi/5} = 5$

Suppose $T = 6$, $\quad$ Possible Reconstruction $n = \frac{t}{T}$

$$x[n] = \cos\left(\frac{\pi}{5} \cdot nT\right) = \cos\left(\frac{6\pi}{5} n\right) \to \cos\left(\frac{6\pi}{5} \cdot \frac{t}{6}\right) = \cos\left(\frac{\pi}{5} t\right)$$

$$\text{V.S.} \quad \cos\left(\frac{6\pi}{5} n\right) = \cos\left(2\pi n - \frac{6\pi}{5} n\right) = \cos\left(\frac{4\pi}{5} n\right) \to \cos\left(\frac{4\pi}{5} \cdot \frac{t}{6}\right)$$

Conclude that sinc interpolation reconstructs $\to$ $= \cos\left(\frac{2\pi}{15} t\right)$

Another possibility

Suppose $T = 2$,

$$x[n] = \cos\left(\frac{\pi}{5} \cdot nT\right) = \cos\left(\frac{2\pi}{5} n\right) \longrightarrow \boxed{\cos\left(\frac{\pi}{5} t\right)}$$

lower freq

$$\text{V.S.} \quad \cos\left(2\pi n - \frac{2\pi}{5} n\right) = \cos\left(\frac{8\pi}{5} n\right) \to \cos\left(\frac{4\pi}{5} t\right)$$

Sinc interpolation reconstructs $\cos \frac{\pi}{5} t$ and is perfect.

$x(t) = \sin(0.2\pi t)$

$= \cos(0.2\pi t - \frac{\pi}{2})$

what T creates an aliased copy

$f(t) = -\sin(\frac{\pi}{15} t)$

$= \cos(\frac{\pi}{15} t + \frac{\pi}{2})$

$x[n] = \cos(0.2\pi nT - \frac{\pi}{2})$

$= \cos(2\pi n - 0.2\pi nT + \frac{\pi}{2})$

because $\cos(2\pi n - x) = \cos(x)$

$= \cos((2\pi - 0.2\pi T)n + \frac{\pi}{2})$

Reconstruct $x(t) = \cos((2\pi - 0.2\pi T)\frac{t}{T} + \frac{\pi}{2})$

$= \cos((\frac{2\pi}{T} - 0.2\pi)t + \frac{\pi}{2})$

Find T s.t. $\frac{\pi}{15} = \frac{2\pi}{T} - 0.2\pi$

$\frac{4\pi}{15} = \frac{2\pi}{T} \rightarrow T = \frac{15}{2} = 7.5$