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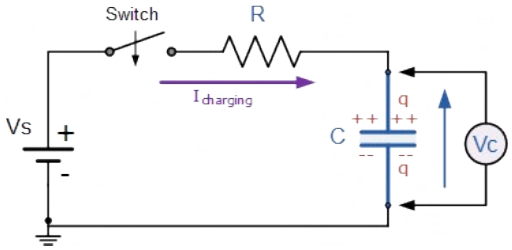
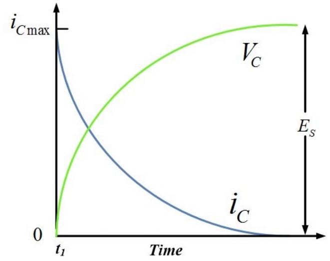
Discussion 2B

Differential Equations & RC Circuits





b

Differential Equations

What are they?

- A differential equation relates a function x(t) and its derivatives.

Examples:

- $\frac{dx}{dt} = \lambda x$
- $x' + 4x = \cos t$
- $x'' + 6x' + 9x = \sin x$
- $x'(t) = 4x(t) + 7$
- $x'' + 6x' + 9x = 0$
- $\frac{d}{dt}\vec{x} = A\vec{x} + \vec{b}$
- $\frac{\partial Q}{\partial t} = c\rho \frac{\partial u}{\partial t}$

The Solution:

- The **solution** of a differential equation is a function x(t) such that the equation is true for all values of t.

Differential Equations

First Order Differential Equations

- A **first order** differential equation only involves a function x(t) and its first derivative x'(t).

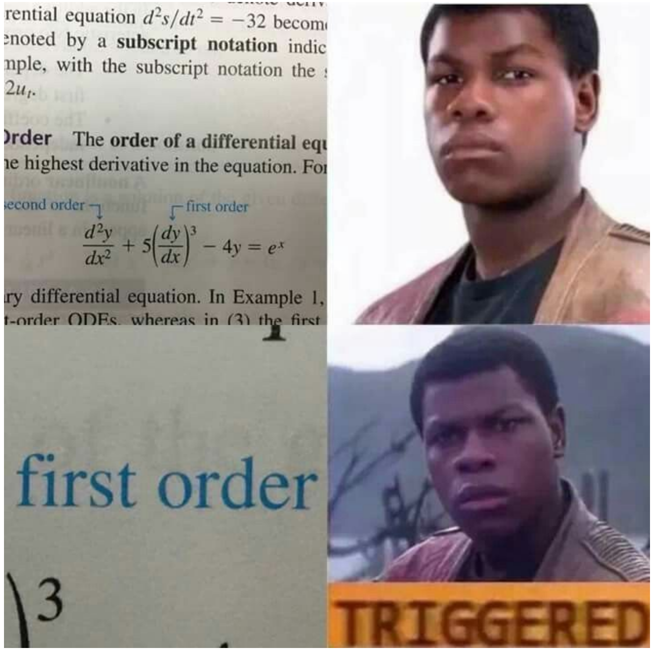
Examples:

$$\frac{dx}{dt} = \lambda x \quad , \quad \frac{dx}{dt} = ax + b \quad , \quad \frac{dx}{dt} + ax = f(t)$$

Initial Conditions

- The **initial conditions** of a differential equation lets us solve for a particular instance of the solution.

$x(0) = x_0$ ← some constant



Two Important Examples

- $x'(t) = 3\,x(t);\ x(0) = 5$

$\lambda = 3$, guess $x = Ke^{3t}$

$x(t=0) = Ke^{3\cdot 0} = Ke^{0} = K = 5$

$x = 5e^{3t}$

- $x'(t) = -2\,x(t) + 4;\ x(0) = 3$

Know: $-2K_1\cancel{e^{-2t}} = -2\cancel{K_1 e^{-2t}} - 2K_2 + 4$

$0 = -2K_2 + 4 \rightarrow K_2 = 2$

$x(t) = K_1 e^{-2t} + 2$

$x(t=0) = K_1 + 2 = 3 \rightarrow K_1 = 1$

$x(t) = e^{-2t} + 2$

$\frac{dx}{dt} = \lambda x\ ;\ x(0) = x_0$

Guess: $x(t) = Ke^{\lambda t}$ ← arbitrary constant

Check: $x'(t) = K\lambda e^{\lambda t}$

$= \lambda Ke^{\lambda t}$

$= \lambda \cdot x(t)$

$\frac{dx}{dt} = \lambda x + u\ ;\ x(0) = x_0$

Try: $\lambda \cancel{Ke^{\lambda t}} = \lambda \cancel{Ke^{\lambda t}} + u$

$0 = u$

contradiction!

Guess: $x(t) = K_1 e^{-2t} + K_2$ $\quad x'(t) = -2x + 4$

$x'(t) = -2K_1 e^{-2t}$

$-2x + 4 = -2(K_1 e^{-2t} + K_2) + 4$

$= -2K_1 e^{-2t} - 2K_2 + 4$

Common Solutions

Let's take a look at the first order differential equation: **x'(t) + a x(t) = b**

diff from $x' = \lambda x + u$

- If b = 0, this is called a **homogeneous** differential equation.
  - Given x(0) = k, the solution is x(t) = $k e^{-at}$

Guess $x(t) = A e^{\lambda t}$

$x(0) = k$

$A = k$

$x(t) = x(0) e^{-at}$

- If b ≠ 0, this is called a **non-homogeneous** differential equation.
  - Given x(0) = k, the solution is x(t) = $x(0) e^{-at} - \frac{b}{a}(e^{-at} - 1)$

Guess: $x(t) = Ae^{\lambda t} + B$

$$= \underbrace{\left(x(0) - \frac{b}{a}\right)}_{A} e^{-at} + \underbrace{\frac{b}{a}}_{B}$$

Uniqueness

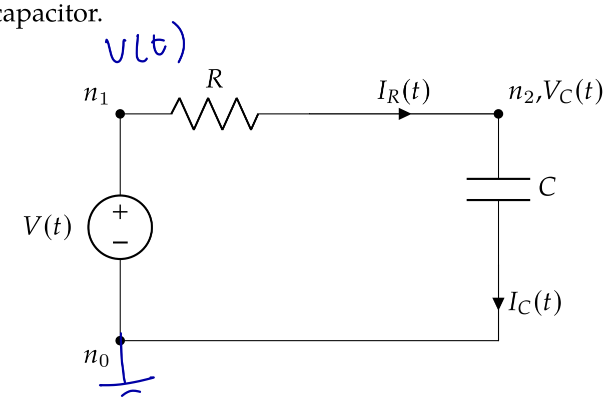
Given a first order differential equation: $\mathbf{x}'(\mathbf{t}) + \mathbf{a}\,\mathbf{x}(\mathbf{t}) = \mathbf{b}$

- If we are given an initial condition $x(0) = k$, then there will always be a unique solution $x(t)$ $= \left(x(0) - \frac{b}{a}\right)e^{-at} + \frac{b}{a}$
- The proof is hard, and we took it out of the HW for this sem.
- When your solution isn't unique ->

1 RC Circuits

In this problem, we will be using differential equations to find the voltage across a capacitor over time in an RC circuit. We set up our problem by first defining three functions over time: $I(t)$ is the current at time t, $V(t)$ is the voltage across the circuit at time t, and $V_C(t)$ is the voltage across the capacitor at time t.

Recall from 16A that the voltage across a resistor is defined as $V_R = RI_R$ where I_R is the current across the resistor. Also, recall that the voltage across a capacitor is defined as $V_C = \frac{Q}{C}$ where Q is the charge across the capacitor.



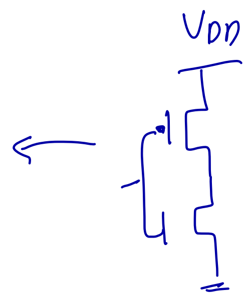


Figure 1: Example Circuit

a) First, find an equation that relates the current through the capacitor $I_C(t)$ with the voltage across the capacitor $V_C(t)$.

Know: $Q = C \cdot V_c$, $I = \frac{dQ}{dt}$

$$\frac{dQ}{dt} = C\frac{dV_c}{dt} \rightarrow I_c = C\frac{dV_c}{dt}$$

b) Using nodal analysis, write a differential equation for the capacitor voltage $V_C(t)$. Note that this is also the voltage for the node n_2.

1. Identify Nodes

Done.

2. Write KCL

$I_R = I_c$

3. I/V Relations

$$I_R = \frac{V(t) - V_c(t)}{R} = C\frac{dV_c}{dt} = I_c$$

$$C\frac{dV_c}{dt} = -\frac{1}{R}V_c(t) + \frac{V(t)}{R}$$

4. Simplify

$$\frac{dV_c}{dt} + \frac{1}{RC}V_c(t) = \frac{V(t)}{RC}$$

c) Let's suppose that at $t = 0$, the capacitor is charged to a voltage V_{DD} ($V_C(0) = V_{DD}$). Let's also assume that $V(t) = 0$ for all $t \geq 0$.

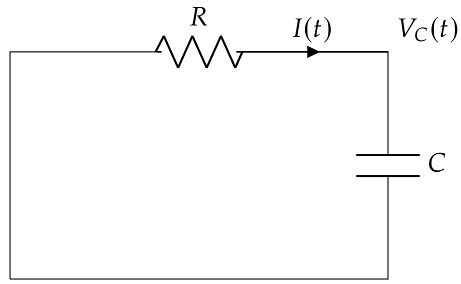


Figure 2: Circuit for part (d)

$V_c(0) = V_{DD}$

$V(t) = 0$ for all $t \geq 0$

Solve the differential equation for $V_C(t)$ for $t \geq 0$.

Know: $\frac{dV_c}{dt} + \underbrace{\frac{1}{RC}}_{a} V_c(t) = \frac{V(t)}{RC} = 0$ $\quad b = 0$

$V_c(0) = V_{DD}$

Given $\frac{dx}{dt} + ax = 0$

$x(t) = x(0)e^{-at}$

Plugging in $a = \frac{1}{RC}$, $x(0) = V_{DD}$, we see that

$V_c(t) = V_{DD}\, e^{-\frac{1}{RC}\cdot t}$

d) Now, let's suppose that we start with an uncharged capacitor $V_C(0) = 0$. We apply some constant voltage $V(t) = V_{DD}$ across the circuit. Solve the differential equation for $V_C(t)$ for $t \geq 0$.

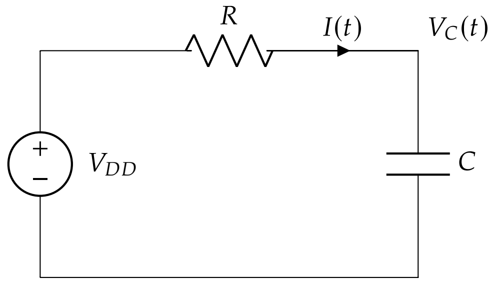


Figure 3: Circuit for part (e)

$V(t) = V_{DD}$ for all t

$$\frac{dV_C}{dt} + \underbrace{\frac{1}{RC}}_{a} V_C(t) = \frac{V(t)}{RC} = \underbrace{\frac{V_{DD}}{RC}}_{b} \qquad V_C(0) = 0$$

$$x(t) = \left(x(0) - \frac{b}{a}\right) e^{-at} + \frac{b}{a}$$

$$= x(0)\, e^{-at} - \frac{b}{a}\left(e^{-at} - 1\right)$$

$$= (0) \cdot e^{-\frac{1}{RC}t} - \frac{\frac{V_{DD}}{RC}}{\frac{1}{RC}}\left(e^{-\frac{1}{RC}t} - 1\right)$$

$$= 0 - V_{DD}\left(e^{-t/RC} - 1\right)$$

$$= V_{DD}\left(1 - e^{-t/RC}\right)$$

Note: There's an alternate method to solving diff-eqs called substitution of vars.

d) Now, let's suppose that we start with an uncharged capacitor $V_C(0) = 0$. We apply some constant voltage $V(t) = V_{DD}$ across the circuit. Solve the differential equation for $V_C(t)$ for $t \geq 0$.

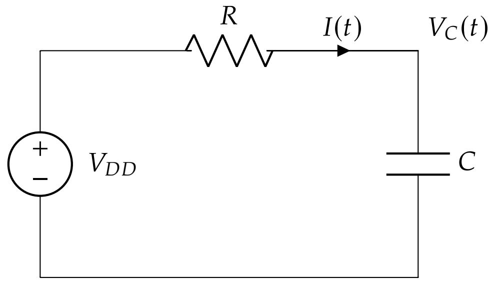


Figure 3: Circuit for part (e)

Alternate Sol:

$$\frac{dV_C}{dt} = -\frac{1}{RC} V_C + \frac{V_{DD}}{RC} \qquad V_C(0) = 0$$

Define a new variable

$$\chi = V_C - V_{DD}$$

Then $\frac{d\chi}{dt} = \frac{dV_C}{dt}$ since V_{DD} is constant

and $-\frac{1}{RC}\chi = -\frac{1}{RC}(V_C - V_{DD})$

plug in χ

This means $\frac{dV_C}{dt} = -\frac{1}{RC} V_C + \frac{V_{DD}}{RC}$ can be rewritten as

$\frac{d\chi}{dt} = -\frac{1}{RC}\chi$. This has solution $\chi(t) = \chi(0)\, e^{-\frac{1}{RC}t}$

$$\chi(0) = V_C(0) - V_{DD} = -V_{DD} \qquad \text{So} \quad \chi(t) = -V_{DD}\, e^{-t/RC}$$

Lastly change variables back to V_C.

$$V_C = \chi + V_{DD} = V_{DD} - V_{DD}\, e^{-t/RC}$$

2 Graphing RC Responses

Consider the following RC Circuit with a single resistor R, capacitor C, and voltage source $V(t)$.

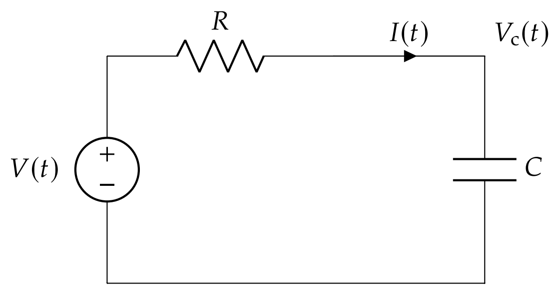
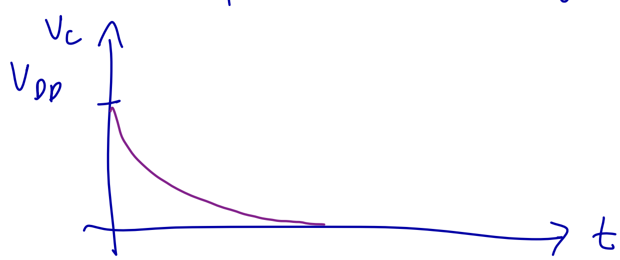


Figure 4: Example Circuit

a) Let's suppose that at $t = 0$, the capacitor is charged to a voltage V_{DD} ($V_c(0) = V_{DD}$) and that $V(t) = 0$ for all $t \geq 0$. Plot the response $V_c(t)$.

Recall that $V_c(t) = V_{DD} e^{-\frac{1}{RC} t}$

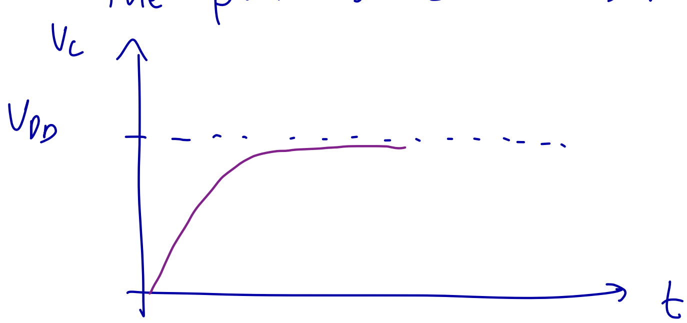
We can plot this on a graphing calculator



b) Now let's suppose that at $t = 0$, the capacitor is uncharged ($V_c(0) = 0$) and that $V(t) = V_{DD}$ for all $t \geq 0$. Plot the response $V_c(t)$.

Recall that $V_c(t) = V_{DD}(1 - e^{-\frac{1}{RC} t})$.

The plot is shown below



To better understand our responses, we now define a **time constant** which is a measure of how long it takes for the capacitor to charge or discharge. Mathematically, we define τ as the time at which $V_C(\tau)$ is $\frac{1}{e} = 36.8\%$ away from its steady state value.

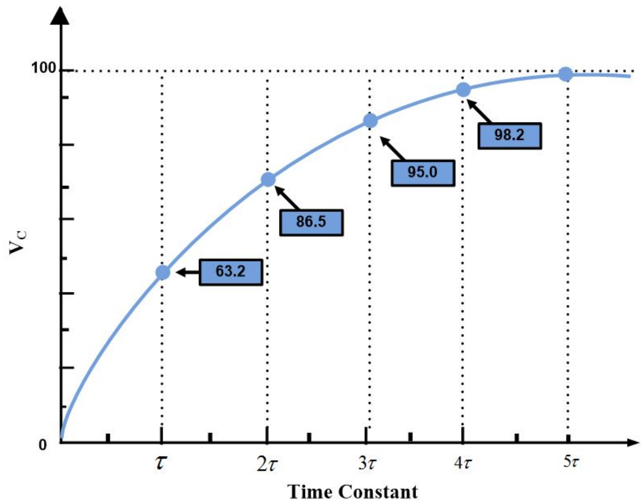
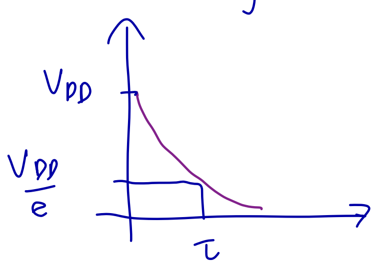


Figure 5: Different values of capacitor voltage at different times, relative to τ.

c) Suppose that $V_{DD} = 5\,\text{V}$, $R = 100\,\Omega$, and $C = 10\,\mu\text{F}$. What is the time constant τ for this circuit?

Let's take the discharging case from part (a).
By definition τ is the time at which $V_C(\tau) = \frac{V_{DD}}{e}$.



We can solve for τ as follows:

$$V_C(\tau) = V_{DD}\, e^{-\tau/RC} = \frac{V_{DD}}{e}$$

$$e^{-\tau/RC} = \frac{1}{e}$$

$$-\tau/RC = \ln\left(\frac{1}{e}\right) = -1 \rightarrow \tau = RC = 1\text{ms}$$

or 0.001s.

d) Going back to part (b), on what order of magnitude of time (nanoseconds, milliseconds, 10's of seconds, etc.) does this circuit settle (V_c is $> 95\%$ of its value as $t \to \infty$)?

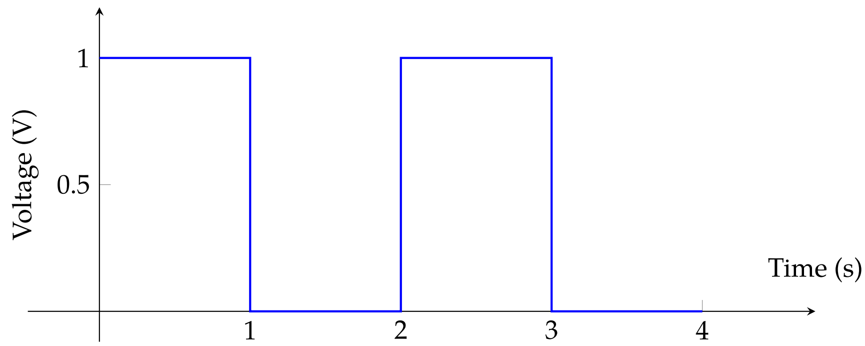
Looking at the graph above, it will take 3τ to reach 95% of the steady state value.

Since $\tau = RC = 1$ms, it will take 3ms to reach within $0.95\,V_{DD}$.

e) Give 2 ways to reduce the settling time of the circuit if we are allowed to change one component in the circuit.

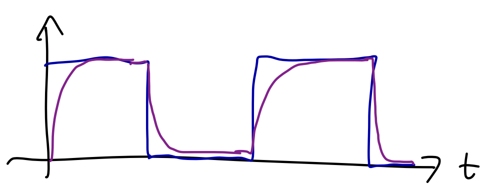
$\tau = RC$. To reduce the time constant, we should either decrease R or decrease C.

f) Suppose we have a source $V(t)$ that alternates between 0 and $V_{DD} = 1\,\text{V}$. Given $RC = 0.1\,\text{s}$, plot the response V_c if $V_c(0) = 0$.



$V(t)$ alternates between 1 and 0. However, it stays constant from $[0,1)$, $[1,2)$, ... Therefore, we can solve the differential equations assume $V(t)$ is constant over an interval.

Plot will look like this



Note that $\tau = 0.1\,s$ and we wait 10τ before switching $V(t)$.

g) Now suppose we have the same source $V(t)$ but $RC = 1\,\text{s}$, plot the response V_c if $V_c(0) = 0$.

Here $\tau = 1\,s$ meaning after 1s we will only reach 63% of V_{DD}.

Note that $63\% = 1 - \frac{1}{e}$.