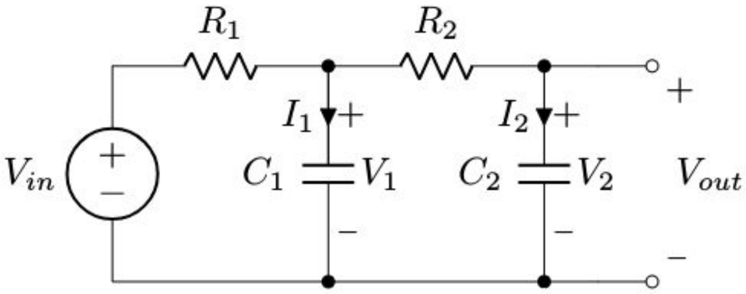
Discussion 3B

Multivariate Differential Equations

Logo credits go to Moses Won



$$\frac{d}{dt}\vec{x} = A\vec{x} + \vec{b}$$

Recap

We currently know how to solve first-order differential equations

- Homogeneous Case: $x'(t) = \lambda\, x(t)$
- Constant Input: $x'(t) = \lambda\, x(t) + u$
- Functional Input: $x'(t) = \lambda\, x(t) + u(t)$ $\leftarrow u(t) = e^{4t}$

$$\int_0^t e^{4\tau} e^{\lambda(t-\tau)}\, d\tau$$

The solution to these differential equations are:

- Homogeneous Case: $x(t) = x(0)\, e^{\lambda t}$
- Constant Input: $x(t) = x(0)\, e^{\lambda t} + \frac{u}{\lambda}\left(e^{\lambda t} - 1\right)$
- Functional Input: $x(t) = x(0)\, e^{\lambda t} + \int_0^t u(\tau)\, e^{\lambda(t-\tau)}\, d\tau$

Vector Differential Equations

A **vector differential equation** is one that involves multiple variables.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$\tilde{x}$ is a vector

Here **x** is a vector and the individual entries $x_1, \ldots, x_n$ are called **states.**

Formally this is called a **state-space** equation, and we'll formalize this in a couple of weeks.

Battle Plan

$$\frac{dx_1}{dt} = 2x_1 + 3x_2$$

$$\frac{dx_2}{dt} = -x_1 + 7x_2$$

$$\frac{d}{dt}\vec{x} = A\vec{x}$$

How do we solve these vector differential equations?

1. Find the eigenvectors and eigenvalues of A.

2. Define a new variable $\mathbf{z} = \mathbf{V}^{-1}\mathbf{x}$ to set-up the differential equation:

$$\frac{d}{dt}\vec{z} = \Lambda\vec{z} \qquad \Lambda: \text{ diagonal} \qquad \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

3. Solve the scalar differential equations $z_i(t)$ using the formulas in Slide 1

$$\frac{d}{dt} z_i = \lambda_i z_i \quad \rightarrow \quad z_i(t) = A e^{\lambda_i t}$$

4. Convert $\mathbf{z}$ back into $\mathbf{x}$ using $\mathbf{x} = \mathbf{Vz}$.

Diagonalization

A matrix **A** is diagonalizable if it has n linearly independent eigenvectors.

Example: A = $\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix}$ is diagonalizable since it has 2 L.I. eigenvectors.

$\lambda_1 = -7 \quad \vec{v}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad \lambda_2 = 6 \quad \vec{v}_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ are Linearly Independent

$B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ B is not diagonalizable

More importantly, we can write $\mathbf{A} = \mathbf{V} \, \mathbf{\Lambda} \, \mathbf{V}^{-1}$. The proof is in Dis 3B Q1

V is a matrix of eigenvectors, Λ is a diagonal matrix of eigenvalues.

Warning: Not every matrix is diagonalizable and there is no relationship between diagonalizability and invertibility.

Dis 3B Q1

Consider a matrix **A** with "eigenpairs:" $(\lambda_1, v_1), \ldots, (\lambda_n, v_n)$

$$V = \begin{bmatrix} | & & | \\ \vec{v}_1 & \ldots & \vec{v}_n \\ | & & | \end{bmatrix} \quad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Show that **AV** = **VΛ:**

Dis 3B Q1

Important matrix trick:

$$A = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix}$$

is a vector

$$A\vec{x} = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1\vec{a}_1 + \ldots + x_n\vec{a}_n$$

Scalars