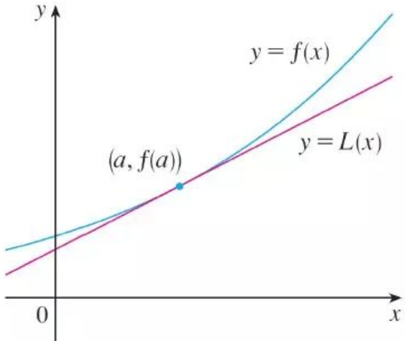
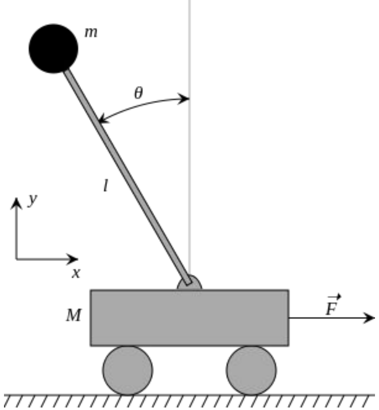
Discussion 7B

Linearization & Stability

Logo credits go to Moses Won





Discussion Feedback

Any feedback is welcome!

- https://forms.gle/HQtVmncbCj2aj69z9

Recap

We've learned how to model state-space equations

- CT Systems were modeled by differential equations.
- DT Systems were modeled by difference equations.

A state-space system is **linear** if it can be written as:

Continuous-Time

$$\frac{d}{dt}\vec{x}(t) = \mathbf{A}\vec{x}(t) + \mathbf{B}\vec{u}(t)$$

eigenvalues $\lambda_1, \ldots, \lambda_n$

Discrete-Time

$$\vec{x}[t+1] = \mathbf{A}\vec{x}[t] + \mathbf{B}\vec{u}[t]$$

Linear systems are "**nice**" since they are predictable and easy to analyze.

Linearization

$\frac{d}{dt}\vec{x} = f(\vec{x}, \vec{u})$

if f is nonlinear

If a system is **nonlinear**, not all hope is lost! **Linearization** is a way to create a linear approximation of the function **f**(**x**, **u**) around an **operating point** (**x***, **u***).

- Note that we are approximating **f**(**x**, **u**) and NOT the function x.

Taylor's Theorem says we can approximate f(x, u) at the point (x*, u*) as:

$$f(x,u) = f(x^*, u^*) + \frac{\partial f}{\partial x}(x^*, u^*) \cdot (x - x^*) + \frac{\partial f}{\partial u}(x^*, u^*) \cdot (u - u^*)$$

scalar func   constant   constants

We can use the **Jacobians** of the function **f** to create a **linear** approximation.

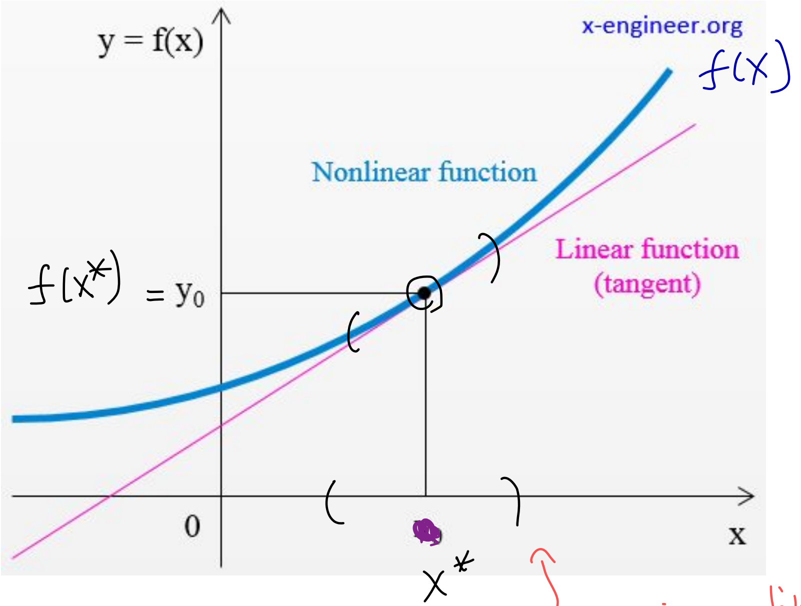
$$f(\vec{x}, \vec{u}) = \underbrace{f(\vec{x}^*, \vec{u}^*)}_{0} + J_{\vec{x}}(x^*, u^*) \cdot (\vec{x} - \vec{x}^*) + J_{\vec{u}}(\vec{x}^*, \vec{u}^*) \cdot (\vec{u} - \vec{u}^*)$$

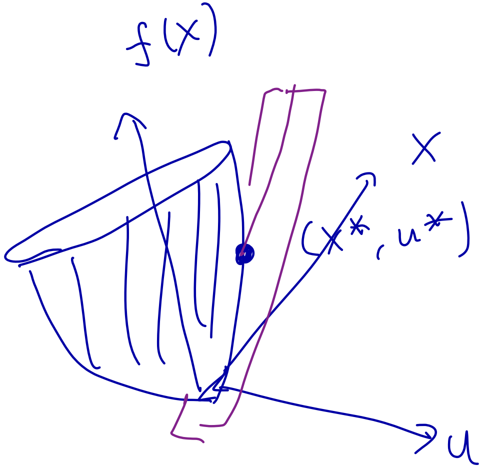
vector func.   matrices

$A \cdot \vec{x}_e + B\,\vec{u}_e + c$ Affine but non-linear

Visualization

$$f(x) = f(x^*) + \frac{\partial f}{\partial x}(x^*) \cdot (x - x^*)$$





region: linear approx is accurate

Equilibrium Points

An **equilibrium point** is a point $(\mathbf{x}^*, \mathbf{u}^*)$ where the system does not change.

Equilibrium points for linear systems can be solved as:

Continuous-Time

$$\frac{d}{dt}\vec{x}(t) = \mathbf{A}\vec{x}(t) + \mathbf{B}\vec{u}(t) = \vec{0}$$

Discrete-Time

$$\vec{x}[t+1] = \mathbf{A}\vec{x}[t] + \mathbf{B}\vec{u}[t] = \vec{x}[t]$$

Equilibrium points of nonlinear systems can be found by solving

$$\frac{d}{dt}\vec{x} = f(\vec{x}, \vec{u}) = \vec{0}$$

$$\vec{x}[t+1] = f(\vec{x}, \vec{u}) = \vec{x}[t]$$

for all t

CT: $\underline{f(\vec{x}^*, \vec{u}^*)} = \vec{0}$

DT: $f(\vec{x}, \vec{u}) = \vec{x}$

Putting Everything Together

To linearize a system, we take the following steps:

1. Find all equilibrium points $(\mathbf{x}^*, \mathbf{u}^*)$ of the function $\mathbf{f}(\mathbf{x}, \mathbf{u})$.
2. Pick one of the equilibrium points to linearize around.
3. Compute the Jacobian matrices J_x and J_u and evaluate them at $(\mathbf{x}^*, \mathbf{u}^*)$.
4. Then linearized system will be of the form:

at non- equilibrium pts

Continuous-Time

$$\frac{d}{dt}\vec{x}_\ell(t) = J_{\vec{x}}\underbrace{\vec{x}_\ell(t)}_{=(\vec{x}-\vec{x}^*)} + J_{\vec{u}}\underbrace{\vec{u}_\ell(t)}_{=(\vec{u}-u^*)} + f(\vec{x}^*, \vec{u}^*)$$

Discrete-Time

$$\vec{x}[t+1] = J_{\vec{x}}\vec{x}_\ell[t] + J_{\vec{u}}\vec{u}_\ell[t]$$

Note: The equilibrium conditions are different for CT / DT Systems but Steps 2 & 3 are identical for CT and DT Systems.

$$\frac{d}{dt}\vec{x} = A\vec{x} + \cancel{B\vec{u}}$$

Stability

A state-space model is **asymptotically stable** if :

- Given zero input (u = 0), the state <u>x(t) converges to 0.</u>

We say a system is **BIBO** (**Bounded Input Bounded Output**) stable if:

- **For every** bounded input u(t), the output x(t) is also bounded.
  - A function f(t) is bounded if |f(t)| < B where B is a finite value.

$$u(t) \longrightarrow \boxed{\text{System}} \xrightarrow{\text{output:}} x(t)$$

Whenever we say **stable** in this class, we are referring to asymptotically stable despite doing all our proofs for BIBO stability.

What you need to know:

A **linear** continuous-time system is **stable** if:

$$\Re e[\lambda_i] < 0 \quad \text{for all } i = 1, \ldots .n$$

$\frac{d}{dt}\vec{x} = A\vec{x}$

$\lambda_1, \ldots, \lambda_n$ eigvals of A

$x_i(t) = a_1 e^{\lambda_1 t} + \ldots + a_n e^{\lambda_n t}$

want them to go to zero

if $Re(\lambda_i) = 0 \rightarrow$ unstable

A **linear** discrete-time system is **stable** if:

$$|\lambda_i| < 1 \quad \text{for all } i = 1, \ldots .n$$

$\vec{x}(t+1) = A\vec{x}(t)$

$x_i(t) = a_1 \lambda_1^t + \ldots + a_n \lambda_n^t$

want these to go to zero

$\left(\frac{1}{2}\right)^t \rightarrow 0$ as $t \rightarrow \infty$

$(2)^t \rightarrow \infty$ as $t \rightarrow \infty$

$|\lambda_i| = 1 \rightarrow$ unstable