



Logo credits go to Moses Won

Discussion 9B

Spectral Theorem & Outer Products

$$\begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 5 & -3 \end{bmatrix}$$

The diagram illustrates the addition of two matrices. A yellow arrow points from the top-left element of the first matrix (3) to the top-left element of the result matrix (7), with the label $3+4=7$ above it. Another yellow arrow points from the top-left element of the second matrix (4) to the top-left element of the result matrix (7), also with the label $3+4=7$ above it.

Diagonal matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Symmetric matrix

$$\begin{bmatrix} 1 & 2 & 6 \\ 2 & 8 & 4 \\ 6 & 4 & 5 \end{bmatrix}$$

The diagram illustrates a symmetric matrix with elements 1, 2, 6, 8, 4, 5. Colored arrows indicate the symmetry: a pink arrow from 2 to 2, a blue arrow from 6 to 6, a red arrow from 4 to 4, a green arrow from 1 to 1, a yellow arrow from 8 to 8, and a purple arrow from 5 to 5.

Recap

New module on Linear Algebra & Machine Learning

- Using Linear Algebra / ML we can **learn** how our systems behave.
- Last time, we looked at **System Identification** which used Least-Squares to learn an unknown state-space model.
- In lecture, you were introduced to the **Singular Value Decomposition** which is a way to break down a matrix as a sum of its “features.”

1 singular value



5 singular values



10 singular values



20 singular values



50 singular values



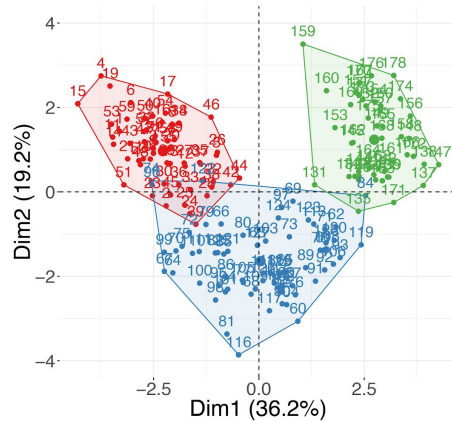
All 499 singular values



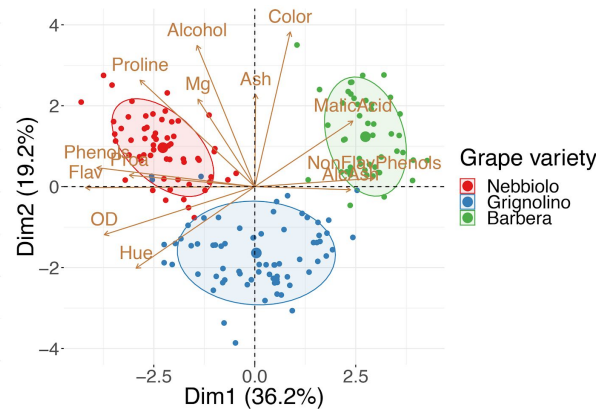
Singular Value Decomposition

The SVD has a lot of applications in Image Processing, ML, Controls, etc.

- It tells us which features of a matrix are the “most important.”
- Used in Data Science to perform dimensionality reduction.
- The SVD is also used in Controls to reach a target with “minimum energy”



(a) Sample Projection



(b) Biplot

Inner Products

Today, we will focus on the Linear Algebra fundamentals that build up to the Singular Value Decomposition.

An **inner product** is a way to multiply two vectors and output a scalar.

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y}$$

$$V = \mathbb{R}^n$$

$$\vec{x}^T \vec{y} \in \mathbb{R}$$

$$1 \times n \quad n \times 1$$

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T A \vec{y}$$

Dot Product

$$\vec{x}, \vec{y} \in \mathbb{R}^n$$

other i.p. \rightarrow

Inner products let us define the **norm** of a vector:

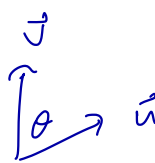
$$\|\vec{x}\| = \sqrt{\vec{x}^T \vec{x}}$$

size of a vector

$$\|\vec{x}\|^2 = \vec{x}^T \vec{x} = \langle \vec{x}, \vec{x} \rangle$$

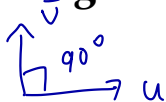
Orthogonality

$$\langle \vec{u}, \vec{v} \rangle = \|\vec{u}\| \|\vec{v}\| \cos \theta = 0$$

$$\cos \theta = 0 \rightarrow \theta = \frac{\pi}{2}$$


Two vectors, \mathbf{u} and \mathbf{v} are **orthogonal** if their inner product is 0.

$$\langle \vec{u}, \vec{v} \rangle = 0$$



A set of vectors $\{\vec{u}_1, \dots, \vec{u}_n\}$ is **orthonormal** if all vectors are mutually orthogonal and have norm 1.

$$\langle \vec{u}_i, \vec{u}_j \rangle = 0$$

if $i \neq j$

$$\|\vec{u}_i\| = 1$$

for all $i = 1, \dots, n$

$$\rightarrow \|\vec{u}_i\|^2 = \langle \vec{u}_i, \vec{u}_i \rangle = 1$$

A square matrix with orthonormal columns is called a **unitary** or **orthonormal** matrix.

$$U = \begin{bmatrix} | & & | \\ \vec{u}_1 & \dots & \vec{u}_n \\ | & & | \end{bmatrix}$$

$$U^T U = I$$

$$U^T = U^{-1}$$

$$\begin{bmatrix} - & \vec{u}_1^T & - \\ & \vdots & \\ - & \vec{u}_n^T & - \end{bmatrix} \begin{bmatrix} | & & | \\ \vec{u}_1 & \dots & \vec{u}_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

Spectral Theorem

A is symmetric if $A = A^T$
 $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

Given an $n \times n$ **symmetric** matrix A , the following statements are true:

1. A has real eigenvalues.
2. A has n linearly independent eigenvectors
 - a. In other words, A is always diagonalizable.
3. The eigenvectors of A can form an orthonormal basis for \mathbb{R}^n .
 - a. This means V is unitary and that $A = V\Lambda V^{-1} = V\Lambda V^T$.

$$\begin{aligned} A &= V\Lambda V^{-1} \\ &= V\Lambda V^T \end{aligned}$$

We can pick orthonormal eigenvectors for the matrix A
 V will be an orthonormal matrix