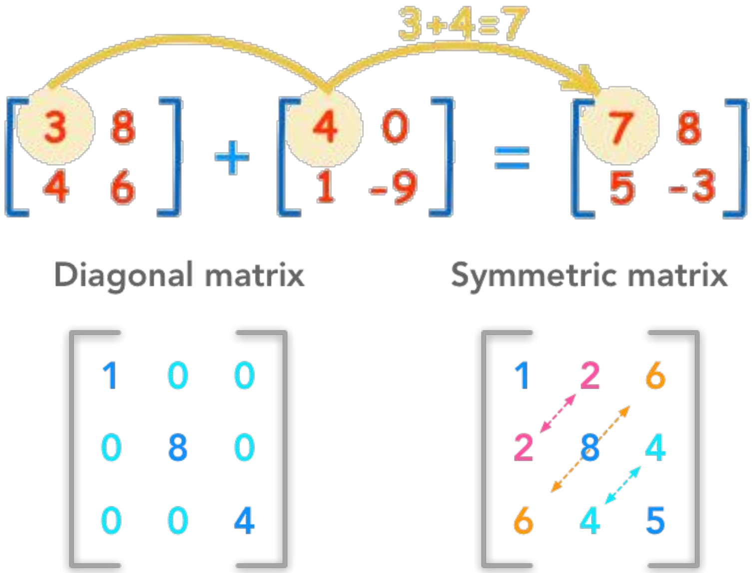
Discussion 9B

Spectral Theorem & Outer Products

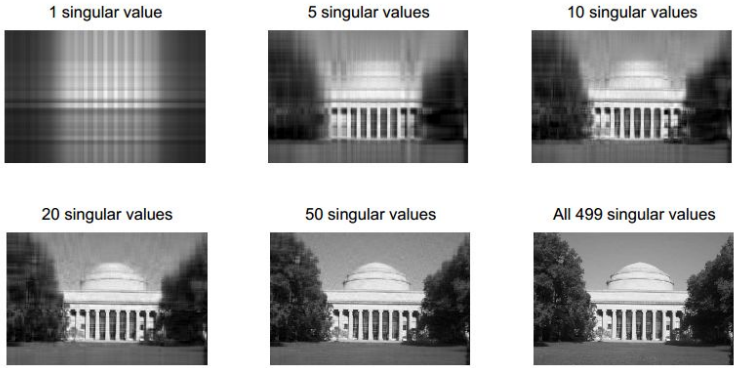
Logo credits go to Moses Won



Recap

New module on Linear Algebra & Machine Learning

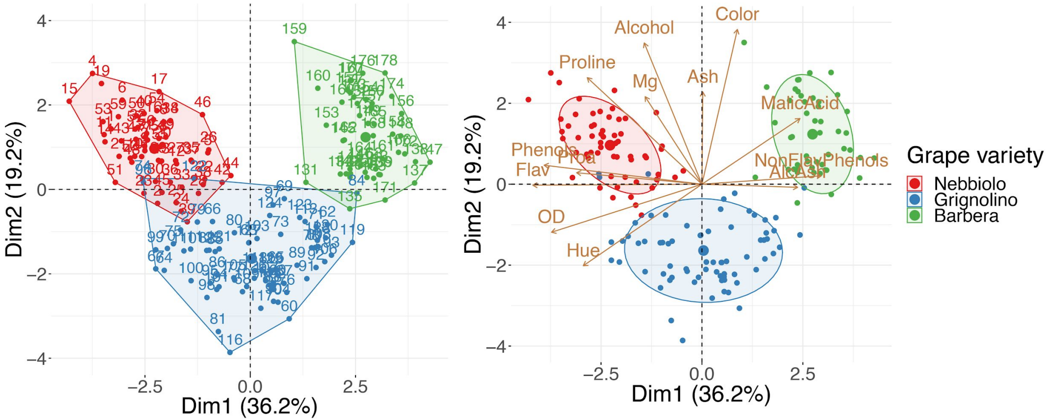
- Using Linear Algebra / ML we can **learn** how our systems behave.
- Last time, we looked at **System Identification** which used Least-Squares to learn an unknown state-space model.
- In lecture, you were introduced to the **Singular Value Decomposition** which is a way to break down a matrix as a sum of its “features.”



Singular Value Decomposition

The SVD has a lot of applications in Image Processing, ML, Controls, etc.

- It tells us which features of a matrix are the “most important.”
- Used in Data Science to perform dimensionality reduction.
- The SVD is also used in Controls to reach a target with “minimum energy”



(a) Sample Projection (b) Biplot

Inner Products

Today, we will focus on the Linear Algebra fundamentals that build up to the Singular Value Decomposition.

An **inner product** is a way to multiply two vectors and output a scalar.

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y}$$

$V = \mathbb{R}^n$

$\vec{x}, \vec{y} \in \mathbb{R}^n$

$\underset{1 \times n}{\vec{x}^T} \; \underset{n \times 1}{\vec{y}} \in \mathbb{R}$

Dot Product

$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T A \vec{y}$

other i.p.

Inner products let us define the **norm** of a vector:

$$||\vec{x}|| = \sqrt{\vec{x}^T \vec{x}}$$

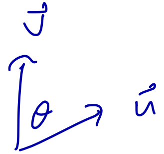
size of a vector

$||\vec{x}||^2 = \vec{x}^T \vec{x} = \langle \vec{x}, \vec{x} \rangle$

Orthogonality

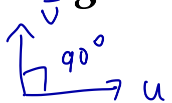
$$\langle \vec{u}, \vec{v} \rangle = \|\vec{u}\|\,\|\vec{v}\| \cos\theta = 0$$

$$\cos\theta = 0 \rightarrow \theta = \frac{\pi}{2}$$



Two vectors, **u** and **v** are **orthogonal** if their inner product is 0.

$$\langle \vec{u}, \vec{v} \rangle = 0$$



A set of vectors $\{\vec{u}_1, \ldots, \vec{u}_n\}$ is **orthonormal** if all vectors are mutually orthogonal and have norm 1.

$$\|u_i\|^2 = \langle \vec{u}_i, \vec{u}_i \rangle = 1$$

$$\langle \vec{u}_i, \vec{u}_j \rangle = 0 \quad \text{if} \quad i \neq j \qquad \|u_i\| = 1 \quad \text{for all } i = 1, \ldots, n$$

A square matrix with orthonormal columns is called a **unitary** or **orthonormal** matrix.

$$U = \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & & | \end{bmatrix}, \quad U^T U = I \quad \begin{bmatrix} - & u_1^T & - \\ & \vdots & \\ - & u_n^T & - \end{bmatrix} \begin{bmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{bmatrix}$$

$$U^T = U^{-1} \qquad = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

Spectral Theorem

A is symmetric if $A = A^T$

$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

Given an nxn **symmetric** matrix A, the following statements are true:

1. A has real eigenvalues.
2. A has n linearly independent eigenvectors
   a. In other words, A is always diagonalizable.
3. The eigenvectors of A can form an orthonormal basis for R^n.
   a. This means **V** is unitary and that $\mathbf{A} = \mathbf{V\Lambda V^{-1}} = \mathbf{V\Lambda V^{T}}$.

$$A = V\Lambda V^{-1}$$
$$= V\Lambda V^T$$

We can pick orthonormal eigenvectors for the matrix A

V will be an orthonormal matrix